

On the form factors of local operators in the lattice sine-Gordon model

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Abstract

We develop a method for computing form factors of local operators in the framework of Sklyanin's separation of variables (SOV) approach to quantum integrable systems. For that purpose, we consider the sine-Gordon model on a finite lattice and in finite dimensional cyclic representations as our main example. We first build our two central tools for computing matrix elements of local operators, namely, a generic determinant formula for the scalar products of states in the SOV framework and the reconstruction of local fields in terms of the separate variables. The general form factors are then obtained as sums of determinants of finite dimensional matrices, their matrix elements being given as weighted sums running over the separate variables and involving the Baxter Q-operator eigenvalues.

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1 Introduction

The computation of general matrix elements and correlation functions of local operators is one of the fundamental problems of quantum field theory and statistical mechanics. These objects contain indeed the key dynamical and measurable quantities of the corresponding physical systems, see e.g. [1–4]. In the integrable (low dimensional) situation [5–11], thanks to the existence of powerful algebraic structures related to the Yang-Baxter algebra (see e.g. [12–20] and references therein), significant progress towards their exact determination has been obtained in the last thirty years. Such results concern in particular models solvable by means of the algebraic Bethe ansatz like the XXZ Heisenberg spin chain [5–9]. They were first obtained at the free fermion point, namely for the Ising model and the Heisenberg chain (for anisotropy Δ equal to zero) [21–24]. Going beyond such a free fermion case involves a deep use of the Yang-Baxter algebra. After historical attempts for the finite chain in the framework of the Bethe ansatz (see [18] and references therein) but leading in fact to implicit representations in terms of dual fields, explicit representations for form factors and correlation functions were first obtained directly in the infinite chain (in the massive regime) [20, 25]. The underlying quantum algebra structure was instrumental there together with a few assumptions on the representation of the Hamiltonian and of the local spin operators within the representation theory of this quantum affine algebra. Similar results for the disordered regime were then derived using the assumption that correlation functions should satisfy q -deformed KZ equations in close analogy with the massive regime [26]. The derivation of these results (both for massive and massless regimes and in the presence of a magnetic field) was later obtained in the framework of the algebraic Bethe ansatz, starting from finite size systems, thanks to the resolution of the quantum inverse scattering problem [27–29]. Further investigations led to the extension of these results to the non-zero temperature case [30, 31] and also to the case of non-trivial (integrable) boundary conditions [32, 33]. The computation of physical (two-point) correlation functions required sophisticated summation techniques of the elementary blocks of correlation functions [34–37] that ultimately led to the exact computation of their asymptotic behavior [38–41]. It is also worth mentioning that controlled numerical summation techniques combined with these exact results on form factors led to the determination of dynamical structure factors in very good agreement with actual neutron scattering experiments on magnetic crystals, see e.g. [42–45]. Another important line of research revealed powerful hidden Grassmann structures that could also be used in the analysis of the conformal limit of these models [46–52]. Finally, it was recently shown how to obtain the asymptotic behavior of the correlation functions for critical systems through their expansion in terms of form factors in the finite volume [53–55].

For more sophisticated systems, like lattice integrable models related to higher rank algebras or for integrable relativistic quantum field theories, the present state of the art is slightly less satisfactory despite considerable efforts.

On the one hand, the bootstrap approach has provided great insights into the structure of the exact scattering matrices [56–62] and form factors of such integrable massive quantum field theories [63–67]. Perturbed conformal field theory¹ (CFT) [73–84] has also been used in this context together with a new understanding of the integrable structure of CFT [85–87]. Several important attempts have also been made to use Sklyanin’s separation of variable method (SOV) [88–94] in these more complicated algebraic situations and in particular in the case of infinite dimensional representations associated to the quantum fields [95–102].

¹ See [68–72] and references therein for some literature on conformal field theories.

One should also mention that the new fermionic structures mentioned above in the case of the XXZ lattice model have been used recently to investigate the structure of matrix elements of the sine-Gordon model in the infinite volume limit [51, 52].

On the other hand, although these advances lead to invaluable informations on these theories, a direct computation of the form factors and correlation functions starting from the description of their local fields and using the given Hamiltonian dynamics is still missing. The main reason is that although these approaches succeeded to describe more and more efficiently both the space of states of such models and the space of their form factors, the connection between the set of operators used to reach this goal and the local operators of the theory one is interested in has not been found yet. Consequently, the matrix elements and correlation functions of these local fields can at best be identified through indirect arguments. In other words, contrary to the above mentioned finite dimensional cases, the solution of the quantum inverse scattering problem is in general not known in the field theory situation; moreover, it appears very often in these more complicated models (either on the lattice or in the continuum) that the usual algebraic Bethe ansatz is no longer applicable due to the lack of a proper reference state, and that the SOV framework [88–90] has to be considered instead. Although this beautiful method is quite general and powerful to describe the spectrum of these models, the problem of reconstructing the local operators of the corresponding theories in terms of the separate variables is in general still open (see however [99]).

To explain in more detail the main issues of this problem, let us consider in particular the sine-Gordon model which will be the main example analyzed in this article. As we just recall, integrable massive field theories in infinite volume can be solved *on-shell*² through the determination of their exact S-matrices [56–62] which completely characterize the particle dynamics. In this particle formulation of the theory a direct access to the local fields is missing and any information about them needs to be extracted from the particle dynamics. The monodromy properties and the singularity structure provide a set of functional equations [63–67] for the form factors of the local fields on asymptotic particle states. These form factor equations are uniquely fixed by the knowledge of the exact S-matrix and the space of their solutions is expected to coincide with that of the local operators of the theory; many results are known on the form factors of local fields, see for example [78–81], [103–113] and references therein for some literature related to sine-Gordon model. However, it is worth pointing out that the form factor equations only contain information on symmetry data of the fields (like charges and spin) and after fixing them to some values, we are still left with an infinite dimensional space of form factor solutions which should correspond to the infinite dimensional space of local fields sharing these data. There is a large literature dedicated to the longstanding problem of the identification of the local fields in the scattering formulation of quantum field theories. Different methods have been introduced to address it; one important line of research is related to the description of massive integrable quantum field theories as (super-renormalizable) perturbations of conformal field theories by relevant local fields [73–84]. This characterization has led to the expectation that the perturbations do not change the structure of the local fields in this way leading to the attempt to classify the local field content of massive theories by that of the corresponding ultraviolet conformal field theories. The latter issue was first addressed in [80] in the simplest massive free theory, namely the Ising model. There a conjecture was introduced defining a correspondence between mild asymptotic behavior at high energy of form factors and chiral local fields. Such a conjecture was justified showing the isomorphism of the space of chiral local fields in the

²See for a review [83] and references therein.

massive and conformal models. The extension of the chiral isomorphism to interacting massive integrable theories was done in [114] for several massive deformations of minimal conformal models, in [115] for the sine-Gordon model and in [116] for all its reductions to unitary minimal models³. The problem to extend the classification also to non-chiral local fields was analyzed in a series of works [124–127]; in particular, for the massive Lee-Yang model, the first proof that the operator space determined by the particle dynamics coincides with that prescribed by conformal symmetry at criticality was given in [127]. While these are indubitably important results on the global structure of the operator space in massive theories it is worth pointing out that they do not lead to the full identification of particular local fields⁴. In [93] a criterion has been introduced based on the quasi-classical characterization of the local fields; it has been fully described in the special cases of the restricted sine-Gordon model at the reflectionless points for chiral fields and verified on the basis of counting arguments [94].

This makes clear that, in the S-matrix formulation of massive quantum integrable theories in infinite volume, the main open problem remains the absence of a direct reconstruction of the local fields. One of our motivation for the present work is to define an exact setup to solve this problem for one of the most paradigmatic integrable quantum field theory, namely the sine-Gordon model. As a first step we will consider the discretized version of the sine-Gordon field theory on a finite lattice. Moreover we will simplify its dynamics by considering the case for which the representation space of the exponents of the field and conjugated momentum is an arbitrary finite dimensional cyclic representation of the quantum algebra, namely the case where the parameter $q = e^{i\hbar}$ is a root of unity. This case is interesting in its own right [128–130] and we also believe that the treatment of the full continuum theory could then be reached by taking the needed limits; at least, we expect to be able to identify the main ingredients and structures necessary to reach this goal and learn enough to extend it to other models on the lattice or in the continuum.

It is worth to describe schematically the microscopic approach that we intend to follow to solve integrable quantum field theories by the complete characterization of their spectrum and of their dynamics.

The first goal is the solution of the spectral problem, for the lattice and the continuum theories:

- i) Solution of the spectral problem for the integrable lattice regularization by the construction of the eigenstates and eigenvalues of the transfer matrix using the SOV method.
- ii) Reformulation of the spectrum in terms of nonlinear integral equations (of thermodynamic Bethe ansatz type) and definition of finite volume quantum field theories in the continuum limit.
- iii) Derivation of the S matrix and particle description of the spectrum in the infinite volume (IR) limit.
- iv) Derivation of the renormalization group fixed point conformal spectrum in the UV limit.

The second goal is the solution of the dynamics along the following steps:

- i) Determinant formulae for the scalar product of states in particular involving transfer matrix eigenstates.
- ii) Reconstruction of the local operators in terms of the quantum separate variables.
- iii) Computation of matrix elements of local operators in the eigenstates basis of the transfer matrix.
- iv) Thermodynamic behavior of the above quantities and computation of the physical correlation functions.

³An important role in these studies has been played by the fermionic representations of the characters, as derived for different classes of rational conformal field theories in [117–123].

⁴Apart for some of them, like the components of the stress energy tensor, which can be characterized by physical prescription [124] and [126].

In this paper, we develop partly this program for the lattice regularization of the sine-Gordon model while the completion of it taking into account the required limits to the continuum theory in finite and infinite volume case will be addressed in future publications. Let us remark that the possibility to apply the SOV method for the discretized version of the sine-Gordon model in finite dimensional cyclic representations was demonstrated recently in [128], hence opening the question of the computations of matrix elements of local fields in a completely controlled (finite dimensional) setting. The first result of the present paper is to show that the scalar products of states can be computed in this case as finite dimensional determinants involving in particular, for eigenstates of the Hamiltonian, the corresponding eigenvalues of the Baxter Q-operator; orthogonality of different eigenstates can be proven directly from these expressions. Further, we will show that the lattice discretization of the local fields of the sine-Gordon model can be reconstructed explicitly in terms of the separate variables. These two ingredients finally lead to the determination of the matrix elements of the exponential of the local fields of the model between arbitrary eigenstates of the Hamiltonian.

This article is organized as follows. In Section 2 we define the sine-Gordon model on a finite lattice in the cyclic representations and recall the main ingredients of the SOV method in that context. In Section 3 we show how to compute the scalar products of states in the SOV representations. The next section is devoted to the reconstruction of the local fields in terms of the separate variables. In Section 5 we use these results to compute the form factors of the local fields in terms of finite size determinants. In the last section we comment on these results and compare them to the existing literature.

2 The sine-Gordon model

We use this section to recall the main results derived in [128, 129] on the spectrum description of the lattice sine-Gordon model.

2.1 Definitions

2.1.1 Classical model

The classical sine-Gordon model can be characterized by the following Hamiltonian density:

$$\mathcal{H}_{SG} \equiv (\partial_x \phi)^2 + \Pi^2 + 8\pi\mu \cos 2\beta\phi \quad (2.1)$$

where the field $\phi(x, t)$ is defined for $(x, t) \in [0, R] \times \mathbb{R}$ with periodic boundary conditions $\phi(x + R, t) = \phi(x, t)$. The dynamics of the model in the Hamiltonian formalism is defined in terms of variables $\phi(x, t)$, $\Pi(x, t)$ with the following Poisson brackets:

$$\{\Pi(x, t), \phi(y, t)\} = 2\pi\delta(x - y). \quad (2.2)$$

The classical integrability of the sine-Gordon model is assured thanks to the representation of the equation of motion by a zero-curvature condition:

$$[\partial_t - V(x, t; \lambda), \partial_x - U(x, t; \lambda)] = 0, \quad (2.3)$$

where, by using the Pauli matrices, we have defined:

$$U = k_1 \sigma_1 \cos \beta \phi + k_2 \sigma_2 \sin \beta \phi - k_3 \sigma_3 \Pi, \quad (2.4)$$

$$V = -k_2 \sigma_1 \cos \beta \phi - k_1 \sigma_2 \sin \beta \phi - k_3 \sigma_3 \partial_x \phi, \quad (2.5)$$

$$k_1 = i\beta (\pi\mu)^{1/2} (\lambda - \lambda^{-1}), \quad k_2 = i\beta (\pi\mu)^{1/2} (\lambda + \lambda^{-1}), \quad k_3 \equiv i\frac{\beta}{2}. \quad (2.6)$$

2.1.2 Quantum lattice regularization

In order to regularize the ultraviolet divergences that arise in the quantization of the model a lattice discretization is introduced. The field variables are discretized according to the standard recipe:

$$\phi_n \equiv \phi(n\Delta), \quad \Pi_n \equiv \Delta \Pi(n\Delta), \quad (2.7)$$

where $\Delta = R/N$ is the lattice spacing. Then, the canonical quantization is defined by imposing that ϕ_n and Π_n are self-adjoint operators satisfying the commutation relations:

$$[\phi_n, \Pi_m] = 2\pi i \delta_{n,m}. \quad (2.8)$$

The quantum lattice regularization of the sine-Gordon model⁵ that we use here goes back to [13, 131, 132] and is related to formulations which have more recently been studied in [133, 134]. Here, the basic operators are the exponentials of the fields and of the conjugate momenta:

$$v_n \equiv e^{-i\beta\phi_n}, \quad u_n \equiv e^{i\beta\Pi_n/2}. \quad (2.9)$$

Then, the commutation relations (2.8) imply that the couples of unitary operators (u_n, v_n) generate N independent Weyl algebras \mathcal{W}_n :

$$u_n v_m = q^{\delta_{nm}} v_m u_n, \quad q \equiv e^{-i\pi\beta^2}, \quad (2.10)$$

for any $n \in \{1, \dots, N\}$. In terms of these basic operators the quantum lattice sine-Gordon model can be characterized by the following Lax matrix⁶:

$$L_n(\lambda) = \kappa_n \begin{pmatrix} u_n(q^{-1/2}v_n\kappa_n + q^{1/2}v_n^{-1}\kappa_n^{-1}) & (\lambda_n v_n - (v_n \lambda_n)^{-1})/i \\ (\lambda_n/v_n - v_n/\lambda_n)/i & u_n^{-1}(q^{1/2}v_n\kappa_n^{-1} + q^{-1/2}v_n^{-1}\kappa_n) \end{pmatrix} \quad (2.11)$$

where $\lambda_n \equiv \lambda/\xi_n$ for any $n \in \{1, \dots, N\}$ with ξ_n and κ_n parameters of the model.

The monodromy matrix $M(\lambda)$ is defined by:

$$M(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \equiv L_N(\lambda) \cdots L_1(\lambda), \quad (2.12)$$

⁵The integrability of the time evolution in the lattice sine-Gordon model is known [131, 135–137] and the most convenient way to formulate it uses the Baxter Q-operators; these operators have been constructed for the closely related chiral Potts model in [138], see [128] for a review and rederivation of these points in the lattice sine-Gordon model.

⁶Here, we make our analysis for the lattice sine-Gordon model leaving the κ_n and ξ_n as free parameters. However, the sine-Gordon model in the continuum limit is reproduced taking suitable limits on these parameters by relating them to the mass μ and the radius R of the compactified space direction of the model.

and satisfies the quadratic commutation relations:

$$R(\lambda/\mu) (M(\lambda) \otimes 1) (1 \otimes M(\mu)) = (1 \otimes M(\mu)) (M(\lambda) \otimes 1) R(\lambda/\mu), \quad (2.13)$$

with the 6-vertex R -matrix:

$$R(\lambda) = \begin{pmatrix} q\lambda - q^{-1}\lambda^{-1} & & & \\ & \lambda - \lambda^{-1} & q - q^{-1} & \\ & q - q^{-1} & \lambda - \lambda^{-1} & \\ & & & q\lambda - q^{-1}\lambda^{-1} \end{pmatrix}. \quad (2.14)$$

The elements of $M(\lambda)$ are the generators of the so-called Yang-Baxter algebra and the commutation relations (2.13) imply the mutual commutativity of the elements of the following one-parameter family of operators:

$$T(\lambda) \equiv \text{tr}_{\mathbb{C}^2} M(\lambda), \quad (2.15)$$

the so-called transfer matrix. In the case of a lattice with N even quantum sites, we can also introduce the operator:

$$\Theta \equiv \prod_{n=1}^N v_n^{(-1)^n}, \quad (2.16)$$

which plays the role of a *grading operator* in the Yang-Baxter algebra:

Proposition 6 of [128] *For the even chain, the charge Θ commutes with the transfer matrix and satisfies the following commutation relations with the generators of the Yang-Baxter algebra:*

$$\Theta C(\lambda) = q C(\lambda) \Theta, \quad [A(\lambda), \Theta] = 0, \quad (2.17)$$

$$B(\lambda) \Theta = q \Theta B(\lambda), \quad [D(\lambda), \Theta] = 0. \quad (2.18)$$

Moreover, the Θ -charge allows to express the asymptotics of the leading generators $A(\lambda)$ and $D(\lambda)$:

$$\lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} A(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\pm 1}}{i} \right) \Theta^{\mp 1}, \quad \lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} D(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\pm 1}}{i} \right) \Theta^{\pm 1} \quad (2.19)$$

and hence the asymptotic behavior of the transfer matrix:

$$\lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} T(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\pm 1}}{i} \right) (\Theta + \Theta^{-1}). \quad (2.20)$$

2.2 Cyclic representations

In the present article, we will consider representations where both v_m and u_n have discrete spectra; in particular, we will restrict our attention to the case in which q is a root of unity:

$$\beta^2 = \frac{p'}{p}, \quad p, p' \in \mathbb{Z}^{>0}, \quad (2.21)$$

with p odd and p' even so that $q^p = 1$. The condition (2.21) implies that the powers p of the generators u_n and v_n are central elements of each Weyl algebra \mathcal{W}_n . In this case, we can associate a p -dimensional linear space R_n to any site n of the chain and we can define on it a cyclic representation of \mathcal{W}_n as follows:

$$v_n |k_n\rangle = v_n q^{k_n} |k_n\rangle, \quad u_n |k_n\rangle = u_n |k_n - 1\rangle, \quad \forall k_n \in \{0, \dots, p-1\}, \quad (2.22)$$

with the cyclicity condition,

$$|k_n + p\rangle = |k_n\rangle. \quad (2.23)$$

The vectors $|k_n\rangle$ define a v_n -eigenbasis of the local space R_n and the parameters v_n and u_n characterize the central elements of the \mathcal{W}_n -representation:

$$v_n^p = v_n^p, \quad u_n^p = u_n^p. \quad (2.24)$$

Here, we require the unitarity of the generators u_n and v_n and the orthonormality of the elements of the v_n -eigenbasis w.r.t. the scalar product introduced in R_n .

Let L_n be the linear space dual of R_n and let $\langle k_n|$ be the elements of the dual basis defined by:

$$\langle k_n | k'_n \rangle = (\langle k_n |, |k'_n \rangle) \equiv \delta_{k_n, k'_n} \quad \forall k_n, k'_n \in \{0, \dots, p-1\}. \quad (2.25)$$

From the unitarity of the generators u_n and v_n , the covectors $\langle k_n|$ define a v_n -eigenbasis in the dual space L_n and the following left representation of Weyl algebra \mathcal{W}_n :

$$\langle k_n | v_n = v_n q^{k_n} \langle k_n |, \quad \langle k_n | u_n = u_n \langle k_n + 1 |, \quad \forall k_n \in \{0, \dots, p-1\}, \quad (2.26)$$

with cyclicity condition:

$$\langle k_n | = \langle k_n + p |. \quad (2.27)$$

In the *left* and *right* linear spaces:

$$\mathcal{L}_N \equiv \bigotimes_{n=1}^N L_n, \quad \mathcal{R}_N \equiv \bigotimes_{n=1}^N R_n, \quad (2.28)$$

the representations of the Weyl algebras \mathcal{W}_n induce cyclic left and right representations of dimension p^N of the monodromy matrix elements, i.e. cyclic representations of the Yang-Baxter algebra. Such representations are characterized by the $4N$ parameters $\kappa = (\kappa_1, \dots, \kappa_N)$, $\xi = (\xi_1, \dots, \xi_N)$, $u = (u_1, \dots, u_N)$ and $v = (v_1, \dots, v_N)$. The unitarity of the Weyl algebra generators u_n and v_n implies that the parameters v and u are pure phases and the following Hermitian conjugation properties of the generators of Yang-Baxter algebra hold:

Lemma 1 of [129] *If the parameters of the representation κ_n^2 and ξ_n^2 are real for any $n = 1, \dots, N$ and satisfy the constraints*

$$\varepsilon \equiv -(\kappa_n \xi_n) / (\kappa_n^* \xi_n^*) \quad \text{is uniform along the chain}, \quad (2.29)$$

then it holds:

$$M(\lambda)^\dagger \equiv \begin{pmatrix} A^\dagger(\lambda) & B^\dagger(\lambda) \\ C^\dagger(\lambda) & D^\dagger(\lambda) \end{pmatrix} = \begin{pmatrix} D(\lambda^*) & C(\varepsilon \lambda^*) \\ B(\varepsilon \lambda^*) & A(\lambda^*) \end{pmatrix}, \quad (2.30)$$

which, in particular, implies the self-adjointness of the transfer matrix $T(\lambda)$ for real λ .

Let us define the average value \mathcal{O} of the elements of the monodromy matrix $M(\lambda)$ as

$$\mathcal{O}(\Lambda) = \prod_{k=1}^p \mathcal{O}(q^k \lambda), \quad \Lambda = \lambda^p, \quad (2.31)$$

where \mathcal{O} can be A , B , C or D and we have to remark that the commutativity of each family of operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ implies that the corresponding average values are functions of Λ , so that $\mathcal{B}(\Lambda)$, $\mathcal{C}(\Lambda)$ are Laurent polynomials of degree $[N]$ while $\mathcal{A}(\Lambda)$, $\mathcal{D}(\Lambda)$ are Laurent polynomials of degree \bar{N} in Λ , where we are using the notations:

$$[N] \equiv N - e_N, \quad \bar{N} \equiv N + e_N - 1, \quad e_N \equiv \begin{cases} 1 & \text{for } N \text{ even,} \\ 0 & \text{for } N \text{ odd.} \end{cases} \quad (2.32)$$

Proposition 2.1.

- a) The average values $\mathcal{A}(\Lambda)$, $\mathcal{B}(\Lambda)$, $\mathcal{C}(\Lambda)$, $\mathcal{D}(\Lambda)$ of the monodromy matrix elements are central elements. Furthermore, they satisfy the following relations:

$$(\mathcal{A}(\Lambda))^* \equiv \mathcal{D}(\Lambda^*), \quad (\mathcal{B}(\Lambda))^* \equiv \epsilon \mathcal{C}(\Lambda^*), \quad (2.33)$$

under complex conjugation in the case of self-adjoint representations.

- b) Let

$$\mathcal{M}(\Lambda) \equiv \begin{pmatrix} \mathcal{A}(\Lambda) & \mathcal{B}(\Lambda) \\ \mathcal{C}(\Lambda) & \mathcal{D}(\Lambda) \end{pmatrix}$$

be the 2×2 matrix whose elements are the average values of the elements of the monodromy matrix $M(\lambda)$, then we have,

$$\mathcal{M}(\Lambda) = \mathcal{L}_N(\Lambda) \mathcal{L}_{N-1}(\Lambda) \dots \mathcal{L}_1(\Lambda),$$

where

$$\mathcal{L}_n(\Lambda) \equiv \begin{pmatrix} q^{p/2} u_n^p (\kappa_{2n}^p v_n^p + v_n^{-p}) & \kappa_n^p (\Lambda v_n^p / \xi_n^p - \xi_n^p / \Lambda v_n^p) / i^p \\ \kappa_n^p (\Lambda / v_n^p \xi_n^p - \xi_n^p v_n^p / \Lambda) / i^p & q^{p/2} u_n^{-p} (\kappa_{2n}^p v_n^{-p} + v_n^p) \end{pmatrix}, \quad (2.34)$$

is the 2×2 matrix whose elements are the average values of the elements of the Lax matrix $L_n(\lambda)$.

A similar statement was first proven in [139]. In the present paper, we will be mainly restricted to the case:

$$u_n = 1, \quad v_n = 1 \quad \text{for } n = 1, \dots, N. \quad (2.35)$$

In these representations it holds:

Lemma 2.1. The power $2p$ of the zeros of $B(\lambda)$ are real numbers with possible complex conjugate couples.

Proof. The previous proposition and the equality

$$\mathcal{C}(\Lambda) = \mathcal{B}(\Lambda), \quad (2.36)$$

which holds for the sine-Gordon representations with $u_n^p = v_n^p = 1$, as proven in [128], imply that

$$(\mathcal{B}(\Lambda))^* = \epsilon \mathcal{B}(\Lambda^*), \quad (2.37)$$

and so the statement of the lemma. \square

2.3 SOV-representations of the Yang-Baxter algebra

According to Sklyanin's method [88–90], a separation of variables (SOV) representation for the spectral problem of the transfer matrix $T(\lambda)$ is defined as a representation where the commutative family of operators $B(\lambda)$ is diagonal. In [128], the following theorem has been shown:

Theorem 1 of [128] *For almost all the values of the parameters κ and ξ of the representation, there exists a SOV representation for the sine-Gordon model, i.e. $B(\lambda)$ is diagonalizable and with simple spectrum.*

The proof of this has been given by a recursive construction of the left cyclic SOV-representations for the sine-Gordon model. Let us recall here the left SOV-representations of the generators of the Yang-Baxter algebra. The Proposition 2.1 fixes the average values of $B(\lambda)$:

$$\mathcal{B}(\Lambda) = \left(\prod_{n=1}^N \frac{\kappa_n}{i} \right)^p Z_N^{\mathbf{e}_N} \prod_{a=1}^{[N]} (\Lambda/Z_a - Z_a/\Lambda) \quad (2.38)$$

in terms of the parameters of the representations. Note that the simplicity of the spectrum of $B(\lambda)$ is equivalent to the requirement $Z_a \neq Z_b$ for any $a \neq b \in \{1, \dots, N - \mathbf{e}_N\}$. Moreover, the reality condition of the polynomial $\mathcal{B}(\Lambda)$, proven in Lemma 2.1, implies that we can chose a N -tupla $\{\eta_1^{(0)}, \dots, \eta_N^{(0)}\}$ of p -roots of $\{Z_1, \dots, Z_N\}$ which satisfy the requirements:

$$\left(\eta_a^{(0)} \right)^2 \in \mathcal{R} \text{ or } \left(\eta_a^{(0)} \right)^2 \notin \mathcal{R} \rightarrow \exists b \in \{1, \dots, N\} \setminus a : \left(\eta_a^{(0)} \right)^2 = \left(\left(\eta_b^{(0)} \right)^2 \right)^* . \quad (2.39)$$

Let $\langle \eta_1^{(k_1)}, \dots, \eta_N^{(k_N)} | \in \mathcal{L}_N$ be the generic element of a basis of left eigenstates of $B(\lambda)$:

$$\langle \mathbf{k} | B(\lambda) = \mathbf{b}_{\mathbf{k}}(\lambda) \langle \mathbf{k} | \text{ with } \mathbf{b}_{\mathbf{k}}(\lambda) \equiv \left(\prod_{n=1}^N \frac{\kappa_n}{i} \right) \eta_N^{(k_N) \mathbf{e}_N} \prod_{a=1}^{[N]} (\lambda/\eta_a^{(k_a)} - \eta_a^{(k_a)}/\lambda), \quad (2.40)$$

where we have used the notation

$$\langle \mathbf{k} | \equiv \langle \eta_1^{(k_1)}, \dots, \eta_N^{(k_N)} |, \quad \eta_a^{(k_a)} \equiv q^{k_a} \eta_a^{(0)} \quad \forall a \in \{1, \dots, N\}, \quad \mathbf{k} \equiv (k_1, \dots, k_N) \in \mathbb{Z}_p^N. \quad (2.41)$$

Then the left action of the other Yang-Baxter generators reads:

$$\langle \mathbf{k} | A(\lambda) = \mathbf{e}_N \frac{\mathbf{b}_{\mathbf{k}}(\lambda)}{\eta_N^{(k_N)}} \left(\frac{\lambda}{\eta_{\mathbf{k},A}} \langle \mathbf{k} | T_N^- - \frac{\eta_{\mathbf{k},A}}{\lambda} \langle \mathbf{k} | T_N^+ \right) + \sum_{a=1}^{[N]} \prod_{b \neq a} \frac{\left(\frac{\lambda}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\lambda} \right)}{\left(\frac{\eta_a^{(k_a)}}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\eta_a^{(k_a)}} \right)} a(\eta_a^{(k_a)}) \langle \mathbf{k} | T_a^-, \quad (2.42)$$

$$\langle \mathbf{k} | D(\lambda) = \mathbf{e}_N \frac{\mathbf{b}_{\mathbf{k}}(\lambda)}{\eta_N^{(k_N)}} \left(\frac{\lambda}{\eta_{\mathbf{k},D}} \langle \mathbf{k} | T_N^+ - \frac{\eta_{\mathbf{k},D}}{\lambda} \langle \mathbf{k} | T_N^- \right) + \sum_{a=1}^{[N]} \prod_{b \neq a} \frac{\left(\frac{\lambda}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\lambda} \right)}{\left(\frac{\eta_a^{(k_a)}}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\eta_a^{(k_a)}} \right)} d(\eta_a^{(k_a)}) \langle \mathbf{k} | T_a^+, \quad (2.43)$$

where $\eta_{\mathbf{k},A}$ and $\eta_{\mathbf{k},D}$ are defined by:

$$\eta_{\mathbf{k},A} \equiv \eta_{\mathbf{k},D} \equiv \frac{\prod_{n=1}^N \xi_n}{\prod_{n=1}^{N-1} \eta_n^{(k_n)}}, \quad (2.44)$$

and the shift operators T_n^\pm are defined by:

$$\langle \eta_1^{(k_1)}, \dots, \eta_n^{(k_n)}, \dots, \eta_N^{(k_N)} | T_n^\pm \equiv \langle \eta_1^{(k_1)}, \dots, \eta_n^{(k_n \pm 1)}, \dots, \eta_N^{(k_N)} |. \quad (2.45)$$

The operator family $C(\lambda)$ is uniquely⁷ defined by the quantum determinant relation:

$$\det_q M(\lambda) \equiv A(\lambda)D(q^{-1}\lambda) - B(\lambda)C(q^{-1}\lambda), \quad (2.46)$$

where $\det_q M(\lambda)$ is a central element⁸ of the Yang-Baxter algebra (2.13) which reads:

$$\det_q M(\lambda) \equiv \prod_{n=1}^N \kappa_n^2 (\lambda/\mu_{n,+} - \mu_{n,+}/\lambda) (\lambda/\mu_{n,-} - \mu_{n,-}/\lambda), \quad (2.47)$$

where $\mu_{n,\pm} \equiv i\kappa_n^{\pm 1} q^{1/2} \xi_n$. In the representations which satisfy (2.35) the coefficients of the SOV-representations can be fixed by introducing the following Laurent polynomials:

$$a(\lambda) = (-i)^N \prod_{n=1}^N \frac{\kappa_n}{\lambda_n} (1 + iq^{-1/2} \lambda_n \kappa_n) (1 + iq^{-1/2} \lambda_n / \kappa_n), \quad d(\lambda) = q^N a(-\lambda q), \quad (2.48)$$

Note that these coefficients are related to the quantum determinant by:

$$\det_q M(\lambda) = a(\lambda) d(\lambda/q). \quad (2.49)$$

Note that for the choice of the parameters $\{\kappa_n\} \in i\mathbb{R}^N$ and $\{\xi_n\} \in \mathbb{R}^N$ the numbers $\kappa \equiv \prod_{n=1}^N \kappa_n / i$ and $\mu_{n,\pm}^p$ are real.

2.4 SOV-characterization of the T-spectrum

Let us denote by Σ_T the set of the eigenvalue functions $t(\lambda)$ of the transfer matrix $T(\lambda)$. From the definitions (2.11) and (2.15), Σ_T is a subset of $\mathbb{R}[\lambda^2, \lambda^{-2}]_{N/2}$ where $\mathbb{R}[x, x^{-1}]_M$ denotes the linear space over the field \mathbb{R} of the *real* Laurent polynomials of degree M in the variable x :

$$(f(x))^* = f(x^*) \quad \forall x \in \mathbb{C} \quad \text{with} \quad f(x) \in \mathbb{R}[x, x^{-1}]_M. \quad (2.50)$$

Note that in the case of N even, the Θ -charge naturally induces the grading $\Sigma_T = \bigcup_{k=0}^{(p-1)/2} \Sigma_T^k$, where:

$$\Sigma_T^k \equiv \left\{ t(\lambda) \in \Sigma_T : \lim_{\log \lambda \rightarrow \mp \infty} \lambda^{\pm N} t(\lambda) = \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\pm 1}}{i} \right) (q^k + q^{-k}) \right\}. \quad (2.51)$$

This simply follows from the asymptotics of $T(\lambda)$ and from its commutativity with Θ .

⁷Note that the operator $B(\lambda)$ is invertible except for λ which coincides with one of its zeros, so in general $C(\lambda)$ is defined by (2.46) just inverting $B(\lambda)$. This is enough to fix in an unique way $C(\lambda)$, as it is a Laurent polynomial of degree $[N]$ in λ .

⁸The centrality of the quantum determinant in the Yang-Baxter algebra was first discovered in [140], see also [141] for an historical note.

In the SOV-representations the spectral problem for $T(\lambda)$ is reduced to the following discrete system of Baxter-like equations in the wave-function $\Psi_t(\mathbf{k}) \equiv \langle \mathbf{k} | t \rangle$ of a T -eigenstate $|t\rangle$:

$$t(\eta_r^{(k_r)})\Psi_t(\mathbf{k}) = a(\eta_r^{(k_r)})\Psi_t(T_r^-(\mathbf{k})) + d(\eta_r^{(k_r)})\Psi_t(T_r^+(\mathbf{k})) \quad \forall r \in \{1, \dots, [N]\}. \quad (2.52)$$

Here, we have denoted by $T_r^\pm(\mathbf{k}) \equiv (k_1, \dots, k_r \pm 1, \dots, k_N)$ and $a(\eta_r^{(k_r)})$ and $d(\eta_r^{(k_r)})$ the coefficients of the SOV-representation as defined in (2.48). In the case of N even, from the asymptotics of $T(\lambda)$ given in (2.20), we have to add to the system (2.52) the following equations in the variable $\eta_N^{(k_N)}$:

$$\Psi_{t,\pm k}(T_N^+(\mathbf{k})) = q^{\mp k}\Psi_{t,\pm k}(\mathbf{k}), \quad (2.53)$$

for the wave function $\Psi_{t,\pm k}(\mathbf{k}) \equiv \langle \mathbf{k} | t_{\pm k} \rangle$, where $|t_{\pm k}\rangle$ is a simultaneous eigenstate of $T(\lambda)$ and Θ corresponding to $t(\lambda) \in \Sigma_T^k$ and Θ -eigenvalue $q^{\pm k}$ with $k \in \{0, \dots, (p-1)/2\}$.

In the paper [129], a complete characterization of the T -spectrum (eigenvalues and eigenstates) has been given in terms of a certain class of polynomial solutions of a given functional equation. Let us recall here these results; to this aim let us introduce the one-parameter family $D(\lambda)$ of $p \times p$ matrices:

$$D(\lambda) \equiv \begin{pmatrix} t(\lambda) & -d(\lambda) & 0 & \dots & 0 & -a(\lambda) \\ -a(q\lambda) & t(q\lambda) & -d(q\lambda) & 0 & \dots & 0 \\ 0 & \ddots & & & & \vdots \\ \vdots & & \dots & & & \vdots \\ \vdots & & & \dots & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \dots & 0 & -a(q^{p-2}\lambda) & t(q^{p-2}\lambda) & -d(q^{p-2}\lambda) \\ -d(q^{p-1}\lambda) & 0 & \dots & 0 & -a(q^{p-1}\lambda) & t(q^{p-1}\lambda) \end{pmatrix} \quad (2.54)$$

where for now $t(\lambda)$ is just a real even Laurent polynomial of degree \bar{N} in λ . Then the determinant of the matrix $D(\lambda)$ is an even Laurent polynomial of maximal degree \bar{N} in $\Lambda \equiv \lambda^p$ and we have the following complete characterization of the transfer matrix spectrum:

Proposition 2.2. *The set Σ_T coincides with the set of all the $t(\lambda) \in \mathbb{R}[\lambda^2, \lambda^{-2}]_{\bar{N}/2}$ solutions of the functional equation:*

$$\det_p D(\Lambda) = 0, \quad \forall \Lambda \in \mathbb{C}. \quad (2.55)$$

Moreover, for N odd the spectrum of $T(\lambda)$ is simple and the eigenstate $|t\rangle$ corresponding to $t(\lambda) \in \Sigma_T$ is characterized by:

$$\Psi_t(\mathbf{k}) \equiv \langle \eta_1^{(k_1)}, \dots, \eta_N^{(k_N)} | t \rangle = \prod_{r=1}^N Q_t(\eta_r^{(k_r)}), \quad (2.56)$$

while for N even the simultaneous spectrum of $T(\lambda)$ and Θ is simple and the eigenstate $|t_{\pm k}\rangle$ corresponding to $t(\lambda) \in \Sigma_T^k$ and Θ -eigenvalue $q^{\pm k}$ with $k \in \{0, \dots, (p-1)/2\}$ is characterized by:

$$\Psi_{t,\pm k}(\mathbf{k}) \equiv \langle \eta_1^{(k_1)}, \dots, \eta_N^{(k_N)} | t_{\pm k} \rangle = (\eta_N^{(k_N)})^{\mp k} \prod_{r=1}^{N-1} Q_t(\eta_r^{(k_r)}). \quad (2.57)$$

Here,

$$Q_t(\lambda) = \lambda^{a_t} \prod_{h=1}^{(p-1)N-(b_t+a_t)} (\lambda_h - \lambda), \quad 0 \leq a_t \leq p-1, \quad 0 \leq b_t \leq (p-1)N, \quad (2.58)$$

$$a_t = 0, \quad b_t = 0 \bmod p, \quad \text{for } N \text{ odd} \quad (2.59)$$

$$a_t = \pm k \bmod p, \quad b_t = \pm k \bmod p, \quad \text{for } N \text{ even and } t(\lambda) \in \Sigma_T^k, \quad (2.60)$$

is the unique (up to quasi-constants) real polynomial solution of the Baxter functional equation:

$$t(\lambda)Q_t(\lambda) = a(\lambda)Q_t(\lambda q^{-1}) + d(\lambda)Q_t(\lambda q) \quad \forall \lambda \in \mathbb{C}, \quad (2.61)$$

corresponding to the given $t(\lambda)$, which has been constructed in terms of the cofactors of the matrix $D(\Lambda)$, in Theorems 2 and 3 of [129].

3 Decomposition of the identity in the SOV-basis

The main goal of this section is to obtain the decomposition of the identity in the SOV basis. This is an important step towards the decomposition of any correlation function in terms of matrix elements of local operators in this basis. In the previous sections we recalled how to define a left SOV basis together with the left action of the Yang-Baxter operators on it. To achieve the identity decomposition we need to define a corresponding right SOV basis.

Traditionally, the right states are constructed from the left ones by means of hermitian conjugation. In the case of hermitian B operators, this procedure would lead to right states that are still eigenstates of B by means of its right action. However, from (2.30), we know that the hermitian conjugate of the B operator is the operator C not equal to B . Hence such a procedure (hermitian conjugation of the left SOV basis) would lead to a right SOV basis with respect to the C operator, namely it would be an eigenstate basis for C . Although such a route could eventually lead to an interesting decomposition of the identity, it would result in a very complicated structure of the scalar product of states (in particular, left and right states would have no simple orthogonality properties); moreover, we would not be able to obtain, at least by obvious means, the coefficients in such an identity decomposition in terms of simple determinants. As a consequence, following this path, the structure of the matrix elements of the local operators would hardly be given in terms of determinants as these are usually obtained as rather simple and localized deformations of the corresponding scalar product formula.

Therefore, our strategy will be here quite different : we will construct the right SOV basis such that it will be an eigenstate basis for the right action of the B operators. Hence it will not be obtained as the hermitian conjugate of the left SOV basis; however, due to the simplicity of the B -spectrum, left states will admit natural orthogonality properties with respect to right states. Hence, this procedure will lead to a decomposition of the identity as a single sum over the SOV basis, with coefficients that are not necessarily positive numbers but that are computable by means of rather simple and universal determinant formula. To show this, we will have to compute the action of left B -eigenstates (covectors) on right B -eigenstates (vectors); up to an overall constant these actions are completely fixed by the left and right SOV-representations of the Yang-Baxter algebras when the gauges in the SOV-representations are chosen.

Let us first define the right SOV-representation with respect to the B operator by the following actions:

$$B(\lambda)|\mathbf{k}\rangle = |\mathbf{k}\rangle b_{\mathbf{k}}(\lambda), \quad (3.1)$$

$$A(\lambda)|\mathbf{k}\rangle = e_N(T_N^+|\mathbf{k}\rangle \frac{\lambda}{\eta_{\mathbf{k},A}} - T_N^-|\mathbf{k}\rangle \frac{\eta_{\mathbf{k},A}}{\lambda}) \frac{b_{\mathbf{k}}(\lambda)}{\eta_N^{(k_N)}} + \sum_{a=1}^{[N]} T_a^+|\mathbf{k}\rangle \prod_{b \neq a} \frac{(\frac{\lambda}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\lambda})}{(\frac{\eta_a^{(k_a)}}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\eta_a^{(k_a)}})} \bar{a}(\eta_a^{(k_a)}), \quad (3.2)$$

$$D(\lambda)|\mathbf{k}\rangle = e_N(T_N^-|\mathbf{k}\rangle \frac{\lambda}{\eta_{\mathbf{k},D}} - T_N^+|\mathbf{k}\rangle \frac{\eta_{\mathbf{k},D}}{\lambda}) \frac{b_{\mathbf{k}}(\lambda)}{\eta_N^{(k_N)}} + \sum_{a=1}^{[N]} T_a^-|\mathbf{k}\rangle \prod_{b \neq a} \frac{(\frac{\lambda}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\lambda})}{(\frac{\eta_a^{(k_a)}}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\eta_a^{(k_a)}})} \bar{d}(\eta_a^{(k_a)}), \quad (3.3)$$

on the generic right B-eigenstate $|\mathbf{k}\rangle \equiv |\eta_1^{(k_1)}, \dots, \eta_N^{(k_N)}\rangle \in \mathcal{R}_N$. Here, the coefficients $\bar{a}(\eta_a)$ and $\bar{d}(\eta_a)$ of the representation are fixed up to the gauge by the condition:

$$\det_q M(\lambda) = \bar{d}(\lambda) \bar{a}(\lambda/q); \quad (3.4)$$

while $C(\lambda)$ is uniquely defined by the quantum determinant relation (2.47).

3.1 Action of left B-eigenstates on right B-eigenstates

It is worth remarking that both for the right and left SOV-representations we are explicitly asking that the coefficients of the representations of $A(\lambda)$ and $D(\lambda)$ depend only on the zeros of $B(\lambda)$ on which the corresponding shift operator acts; i.e. they are separated w.r.t. the zeros of $B(\lambda)$. Naturally such requirement is compatible with the Yang-Baxter algebra and the quantum determinant relation but it is not implied by them. The main point that we are going to prove here is that this separation requirement completely fixes the actions of the generic covector in the left SOV-basis on the generic vector in the right SOV-basis. It may be helpful to explain this last statement in terms of the properties of the matrices which define the change of basis to the SOV-basis. Let us define the following isomorphism:

$$\varkappa : (h_1, \dots, h_N) \in \{1, \dots, p\}^N \rightarrow j = \varkappa(h_1, \dots, h_N) \equiv h_1 + \sum_{a=2}^N p^{(a-1)}(h_a - 1) \in \{1, \dots, p^N\}, \quad (3.5)$$

then we can write:

$$\langle \mathbf{y}_j | = \langle \mathbf{x}_j | U^{(L)} = \sum_{i=1}^{p^N} U_{j,i}^{(L)} \langle \mathbf{x}_i | \quad \text{and} \quad |\mathbf{y}_j\rangle = U^{(R)} |\mathbf{x}_j\rangle = \sum_{i=1}^{p^N} U_{i,j}^{(R)} |\mathbf{x}_i\rangle, \quad (3.6)$$

where we have used the notations:

$$\langle \mathbf{y}_j | \equiv \langle \eta_1^{(h_1)}, \dots, \eta_N^{(h_N)} | \quad \text{and} \quad |\mathbf{y}_j\rangle \equiv |\eta_1^{(h_1)}, \dots, \eta_N^{(h_N)}\rangle, \text{ for } j = \varkappa(h_1, \dots, h_N), \quad (3.7)$$

to represent, respectively, the states of the left and right SOV-basis and:

$$\langle \mathbf{x}_j | \equiv \otimes_{n=1}^N \langle h_n | \quad \text{and} \quad | \mathbf{x}_j \rangle \equiv \otimes_{n=1}^N | h_n \rangle, \text{ for } j = \varkappa(h_1, \dots, h_N), \quad (3.8)$$

to represent, respectively, the states of the left and right original v_n -orthonormal basis. Here, $U^{(L)}$ and $U^{(R)}$ are the $p^N \times p^N$ matrices for which it holds:

$$U^{(L)} B(\lambda) = \Delta_B(\lambda) U^{(L)}, \quad B(\lambda) U^{(R)} = U^{(R)} \Delta_B(\lambda), \quad (3.9)$$

where $\Delta_B(\lambda)$ is a diagonal $p^N \times p^N$ matrix. The diagonalizability and simplicity of the B-spectrum imply the invertibility of the matrices $U^{(L)}$ and $U^{(R)}$ and the fact that all the diagonal entries of $\Delta_B(\lambda)$ are Laurent polynomials in λ with different zeros. Then the following proposition holds:

Proposition 3.1. *Right and left SOV-basis are right and left B-eigenbasis such that the $p^N \times p^N$ matrix:*

$$M \equiv U^{(L)} U^{(R)} \quad (3.10)$$

is diagonal and characterized by:

$$M_{jj} = \langle \mathbf{y}_j | \mathbf{y}_j \rangle = \langle \eta_1^{(h_1)}, \dots, \eta_N^{(h_N)} | \eta_1^{(h_1)}, \dots, \eta_N^{(h_N)} \rangle = \frac{C_N \prod_{a=1}^{[N]} \prod_{k_a=1}^{h_a} a(\eta_a^{(k_a)}) / \bar{a}(\eta_a^{(k_a-1)})}{\prod_{1 \leq b < a \leq [N]} (\eta_a^{(h_a)} / \eta_b^{(h_b)} - \eta_b^{(h_b)} / \eta_a^{(h_a)})}, \quad (3.11)$$

where we have denoted:

$$j = \varkappa(h_1, \dots, h_N) \quad (3.12)$$

and C_N is a constant characteristic of the chosen representations.

Proof. The fact that the matrix M is diagonal is a trivial consequence of the orthogonality of left and right eigenstates corresponding to different eigenvalue of $B(\lambda)$ and of the simplicity of the B-spectrum. Indeed, defined $i = \varkappa(k_1, \dots, k_N)$ and $j = \varkappa(h_1, \dots, h_N)$, the following identities hold:

$$b_k(\lambda) M_{ij} = \langle \eta_1^{(k_1)}, \dots, \eta_N^{(k_N)} | B(\lambda) | \eta_1^{(h_1)}, \dots, \eta_N^{(h_N)} \rangle, \quad (3.13)$$

$$b_h(\lambda) M_{ij} = \langle \eta_1^{(k_1)}, \dots, \eta_N^{(k_N)} | B(\lambda) | \eta_1^{(h_1)}, \dots, \eta_N^{(h_N)} \rangle, \quad (3.14)$$

where the identity (3.13) follows acting with $B(\lambda)$ on the left while the identity (3.14) follows acting with $B(\lambda)$ on the right. Then in the case $(k_1, \dots, k_N) \neq (h_1, \dots, h_N)$ the identities (3.13) and (3.14) imply that $M_{ij} = 0$ for any $i \neq j \in \{1, \dots, p^N\}$.

Now, let us compute the matrix element $\theta_a \equiv \langle \eta_1^{(0)}, \dots, \eta_a^{(1)}, \dots, \eta_N^{(0)} | A(\eta_a^{(1)}) | \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} \rangle$, where $a \in \{1, \dots, [N]\}$. Then using the left action of the operator $A(\bar{\eta}_a)$ we get:

$$\theta_a = a(\eta_a^{(1)}) \langle \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} \rangle \quad (3.15)$$

while using the right action of the operator $A(\bar{\eta}_a)$ and the orthogonality of right and left B-eigenstates corresponding to different eigenvalues we get:

$$\theta_a = \prod_{b \neq a, b=1}^{[N]} \frac{(\eta_a^{(1)} / \eta_b^{(0)} - \eta_b^{(0)} / \eta_a^{(1)})}{(\eta_a^{(0)} / \eta_b^{(0)} - \eta_b^{(0)} / \eta_a^{(0)})} \bar{a}(\eta_a^{(0)}) \langle \eta_1^{(0)}, \dots, \eta_a^{(1)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \dots, \eta_a^{(1)}, \dots, \eta_N^{(0)} \rangle \quad (3.16)$$

and so:

$$\frac{\langle \eta_1^{(0)}, \dots, \eta_a^{(1)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \dots, \eta_a^{(1)}, \dots, \eta_N^{(0)} \rangle}{\langle \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} \rangle} = \frac{a(\eta_a^{(1)})}{\bar{a}(\eta_a^{(0)})} \prod_{b \neq a, b=1}^{[N]} \frac{(\eta_a^{(0)}/\eta_b^{(0)} - \eta_b^{(0)}/\eta_a^{(0)})}{(\eta_a^{(1)}/\eta_b^{(0)} - \eta_b^{(1)}/\eta_a^{(0)})}. \quad (3.17)$$

Then by applying (3.17) h_a times, we get:

$$\frac{\langle \eta_1^{(0)}, \dots, \eta_a^{(h_a)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \dots, \eta_a^{(h_a)}, \dots, \eta_N^{(0)} \rangle}{\langle \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} \rangle} = \prod_{k_a=1}^{h_a} \frac{a(\eta_a^{(k_a)})}{\bar{a}(\eta_a^{(k_a-1)})} \prod_{b \neq a, b=1}^{[N]} \frac{(\eta_a^{(0)}/\eta_b^{(0)} - \eta_b^{(0)}/\eta_a^{(0)})}{(\eta_a^{(h_a)}/\eta_b^{(0)} - \eta_b^{(h_a)}/\eta_a^{(0)})}. \quad (3.18)$$

Now, let us consider explicitly the case of even N . In this case we still have to define the recurrence for $a = N$. We compute the matrix element:

$$\theta_N(\lambda) \equiv \langle \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(1)} | A(\lambda) | \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(0)} \rangle, \quad (3.19)$$

acting with $A(\lambda)$ on the right we get:

$$\theta_N(\lambda) \equiv \frac{\lambda \mathbf{b}_k(\lambda)}{\eta_N^{(k_N)} \eta_{k,A}} \langle \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(1)} | \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(1)} \rangle, \quad (3.20)$$

while acting on the left we get:

$$\theta_N(\lambda) \equiv \frac{\lambda \mathbf{b}_k(\lambda)}{\eta_N^{(k_N)} \eta_{k,A}} \langle \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(1)} | \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(1)} \rangle, \quad (3.21)$$

which simply implies:

$$\langle \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(1)} | \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(1)} \rangle = \langle \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(0)} | \eta_1^{(0)}, \dots, \eta_{N-1}^{(0)}, \eta_N^{(0)} \rangle. \quad (3.22)$$

The previous formula implies, for both N even and odd:

$$\frac{\langle \eta_1^{(h_a)}, \dots, \eta_a^{(h_a)}, \dots, \eta_N^{(h_N)} | \eta_1^{(h_a)}, \dots, \eta_a^{(h_a)}, \dots, \eta_N^{(h_N)} \rangle}{\langle \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \dots, \eta_a^{(0)}, \dots, \eta_N^{(0)} \rangle} = \prod_{a=1}^{[N]} \prod_{k_a=1}^{h_a} \frac{a(\eta_a^{(k_a)})}{\bar{a}(\eta_a^{(k_a-1)})} \prod_{1 \leq b < a \leq [N]} \frac{(\frac{\eta_a^{(0)}}{\eta_b^{(0)}} - \frac{\eta_b^{(0)}}{\eta_a^{(0)}})}{(\frac{\eta_a^{(k_a)}}{\eta_b^{(k_b)}} - \frac{\eta_b^{(k_b)}}{\eta_a^{(k_a)}})}. \quad (3.23)$$

To prove it let us observe that (3.18) coincides with (3.23) for $h_2 = \dots = h_N = 0$. Now in the case $h_3 = \dots = h_N = 0$ with $h_1 \neq 0$ and $h_2 \neq 0$, we get (3.23) by the following factorization:

$$\begin{aligned} \frac{\langle \eta_1^{(h_1)}, \eta_2^{(h_2)}, \dots, \eta_N^{(h_N)} | \eta_1^{(h_1)}, \eta_2^{(h_2)}, \dots, \eta_N^{(h_N)} \rangle}{\langle \eta_1^{(0)}, \eta_2^{(0)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \eta_2^{(0)}, \dots, \eta_N^{(0)} \rangle} &= \frac{\langle \eta_1^{(h_1)}, \eta_2^{(0)}, \dots, \eta_N^{(h_N)} | \eta_1^{(h_1)}, \eta_2^{(0)}, \dots, \eta_N^{(h_N)} \rangle}{\langle \eta_1^{(0)}, \eta_2^{(0)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \eta_2^{(0)}, \dots, \eta_N^{(0)} \rangle} \\ &\times \frac{\langle \eta_1^{(h_1)}, \eta_2^{(h_2)}, \dots, \eta_N^{(h_N)} | \eta_1^{(h_1)}, \eta_2^{(h_2)}, \dots, \eta_N^{(h_N)} \rangle}{\langle \eta_1^{(h_1)}, \eta_2^{(0)}, \dots, \eta_N^{(0)} | \eta_1^{(h_1)}, \eta_2^{(0)}, \dots, \eta_N^{(0)} \rangle} \\ &= \prod_{a=1}^2 \prod_{k_a=1}^{h_a} \frac{a(\eta_a^{(k_a)})}{\bar{a}(\eta_a^{(k_a-1)})} \prod_{b=2}^{[N]} \frac{(\eta_1^{(0)}/\eta_b^{(0)} - \eta_b^{(0)}/\eta_1^{(0)})}{(\eta_1^{(h_1)}/\eta_b^{(0)} - \eta_b^{(0)}/\eta_1^{(h_1)})} \\ &\times \frac{(\eta_2^{(0)}/\eta_1^{(h_1)} - \eta_1^{(h_1)}/\eta_2^{(0)})}{(\eta_2^{(h_2)}/\eta_1^{(h_1)} - \eta_1^{(h_1)}/\eta_2^{(h_2)})} \prod_{b=3}^{[N]} \frac{(\eta_2^{(0)}/\eta_b^{(0)} - \eta_b^{(0)}/\eta_2^{(0)})}{(\eta_2^{(h_2)}/\eta_b^{(0)} - \eta_b^{(0)}/\eta_2^{(h_2)})}, \end{aligned} \quad (3.24)$$

and so on for the generic case. Finally, from (3.23) the statement of the proposition follows being by definition:

$$\frac{M_{ji}}{\langle \mathbf{y}_{p^N} | \mathbf{y}_{p^N} \rangle} \equiv \delta_{i,j} \frac{\langle \eta_1^{(h_1)}, \eta_2^{(h_2)}, \dots, \eta_N^{(h_N)} | \eta_1^{(h_1)}, \eta_2^{(h_2)}, \dots, \eta_N^{(h_N)} \rangle}{\langle \eta_1^{(0)}, \eta_2^{(0)}, \dots, \eta_N^{(0)} | \eta_1^{(0)}, \eta_2^{(0)}, \dots, \eta_N^{(0)} \rangle}. \quad (3.25)$$

□

Remark 2. The diagonal matrix M is mainly fixed by the requirement that the left and right representations have a separate form in the zeros of $B(\lambda)$. Indeed, this fixes completely the denominator in all the entries of M . Moreover, the constant C_N is a function of the representation only via the central elements (Z_1, \dots, Z_N) . In the following, we will fix:

$$C_N \equiv (\eta_N^{(0)} p^{1/2})^{\mathbf{e}_N}, \quad (3.26)$$

this choice just amounts to an overall renormalization of the states. Finally, the products of $a(\lambda)/\bar{a}(\lambda q^{-1})$ in the numerator of each matrix element is fixed from the choice of the gauge done in the SOV-representations. Note that we are always free to take the following gauge:

$$\bar{a}(\lambda q^{-1}) \equiv a(\lambda), \quad (3.27)$$

for which the numerator in (3.11) is C_N .

3.2 SOV-decomposition of the identity

The previous results allow to write the following spectral decomposition of the identity \mathbb{I} :

$$\mathbb{I} \equiv \sum_{i=1}^{p^N} \mu_i |\mathbf{y}_i\rangle \langle \mathbf{y}_i|, \quad (3.28)$$

in terms of the left and right SOV-basis. Here,

$$\mu_i \equiv \frac{1}{\langle \mathbf{y}_i | \mathbf{y}_i \rangle}, \quad (3.29)$$

is the required analogue of Sklyanin's *measure*⁹, which is discrete for the cyclic representations of the sine-Gordon model. Explicitly, the SOV-decomposition of the identity reads:

$$\mathbb{I} \equiv \sum_{h_1, \dots, h_N=1}^p \prod_{1 \leq b < a \leq [N]} ((\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2) \frac{|\eta_1^{(h_1)}, \dots, \eta_N^{(h_N)}\rangle \langle \eta_1^{(h_1)}, \dots, \eta_N^{(h_N)}|}{C_N \prod_{b=1}^{[N]} \omega_b(\eta_b^{(h_b)})}, \quad (3.30)$$

where the separate functions $\omega_b(\eta_b^{(h_b)})$ reproduce the numerator in (3.11):

$$\omega_b(\eta_b^{(h_b)}) \equiv \left(\eta_b^{(h_b)} \right)^{[N]-1} \prod_{k_b=1}^{h_b} a(\eta_b^{(k_b)}) / \bar{a}(\eta_b^{(k_b-1)}) \quad (3.31)$$

⁹Note that here it cannot be called a measure as the $B(\lambda)$ operator is not self-adjoint, meaning that in general the μ_i are not positive real numbers. However the above procedure indeed defines a proper decomposition of the identity operator with simple and computable coefficients.

and they are gauge dependent parameters.

Remark 3. Sklyanin's measure¹⁰ has been first introduced by Sklyanin in his article on quantum Toda chain [88]. There, it has been derived as a consequence of the self-adjointness of the transfer matrix w.r.t. the scalar product. In particular, the Hermitian properties of the operator zeros and their conjugate shift operators have been fixed to assure the self-adjointness of the transfer matrix. In the similar but more involved sinh-Gordon model [100], the problem related to the uniqueness of the definition of this measure has been analyzed. There, it has been proven that the measure is in fact uniquely determined once the positive self-adjointness of the generators $A(\lambda)$ and $D(\lambda)$ is required. In the compact case of the sine-Gordon model the method used here insures that the analogue of Sklyanin's measure is uniquely determined up to an overall constant and the choice of gauge, as discussed in the previous remark. Let us mention that the Sklyanin's measure has already been derived in [143] for cyclic representations of the related τ_2 -model¹¹ [147–149]. There the measure has been obtained by a different approach, i.e. by a recursive construction which needs to go through the recursion in the construction of left and right SOV-basis. It is interesting to remark that in our purely algebraic derivation we skip these model dependent features so that the approach we used is suitable for general compact SOV-representations of the 6-vertex Yang-Baxter algebra.

3.3 SOV-representation of left and right T-eigenstates

Up to an overall normalization, the SOV-decomposition of the identity and the SOV-characterization of the transfer matrix spectrum lead to the following representations of the right eigenstate of the transfer matrix $T(\lambda)$:

$$\begin{aligned} |t_{\pm k}\rangle &= \sum_{i=1}^{p^N} \mu_i \langle \mathbf{y}_i | t_{\pm k} \rangle | \mathbf{y}_i \rangle \\ &= \sum_{h_1, \dots, h_N=1}^p \left(\frac{q^{\mp k h_N}}{p^{1/2}} \right)^{\mathbf{e}_N} \prod_{a=1}^{[N]} Q_t(\eta_a^{(h_a)}) \prod_{1 \leq b < a \leq [N]} ((\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2) \frac{|\eta_1^{(h_1)}, \dots, \eta_N^{(h_N)}\rangle}{\prod_{b=1}^{[N]} \omega_b(\eta_b^{(h_b)})}, \end{aligned} \quad (3.32)$$

corresponding to the eigenvalue $t(\lambda) \in \Sigma_T^k$, where the index $k \in \{0, \dots, (p-1)/2\}$ appears only for N even and indicates that $|t_{\pm k}\rangle$ is also a Θ -eigenstate with eigenvalue $q^{\pm k}$. Here $Q_t(\lambda)$ is the solution of the Baxter equation (3.34) defined in Proposition 2.2. In a similar way we can prove that, up to an overall normalization, the left T-eigenstate has the following SOV-representation:

$$\begin{aligned} \langle t_{\pm k} | &= \sum_{i=1}^{p^N} \mu_i \langle t_{\pm k} | \mathbf{y}_i \rangle \langle \mathbf{y}_i | \\ &= \sum_{h_1, \dots, h_N=1}^p \left(\frac{q^{\pm k h_N}}{p^{1/2}} \right)^{\mathbf{e}_N} \prod_{a=1}^{[N]} \bar{Q}_t(\eta_a^{(h_a)}) \prod_{1 \leq b < a \leq [N]} (\eta_a^{(h_a)} - \eta_b^{(h_b)}) \frac{\langle \eta_1^{(h_1)}, \dots, \eta_N^{(h_N)} |}{\prod_{b=1}^{[N]} \omega_b(\eta_b^{(h_b)})}, \end{aligned} \quad (3.33)$$

where $\bar{Q}_t(\lambda)$ is the unique (up to quasi-constants) polynomial solution of the Baxter functional equation:

$$t(\lambda) \bar{Q}_t(\lambda) = \bar{d}(\lambda) \bar{Q}_t(\lambda q^{-1}) + \bar{a}(\lambda) \bar{Q}_t(\lambda q). \quad (3.34)$$

¹⁰See also [97] for further discussions on the measure.

¹¹The SOV analysis for these representations has been first developed in [144].

Remark 4. In the gauge fixed by (3.27), this Baxter equation reads:

$$t(\lambda)\bar{Q}_t(-\lambda) = q^N a(\lambda)\bar{Q}_t(-\lambda q^{-1}) + q^{-N} d(\lambda)\bar{Q}_t(-\lambda q), \quad (3.35)$$

and so, up to quasi-constants, we have:

$$\bar{Q}_t(\lambda) = \lambda^{\chi_N} Q_t(-\lambda), \quad \chi_N = N \bmod p, \quad 0 \leq \chi_N \leq p-1. \quad (3.36)$$

4 Decomposition of the identity in the T-eigenbasis

Here we use the results of the previous section to compute the action of covectors on vectors which in the left and right SOV-basis have a *separate form* similar to that of the transfer matrix eigenstates.

4.1 Action of left separate states on right separate states

Let us start giving the following definition of a left $\langle \alpha_k |$ and a right $|\beta_k\rangle$ separate states; they are respectively a covector and a vector which have the following *separate form* in terms of the SOV-decomposition of the identity:

$$\langle \alpha_k | = \sum_{h_1, \dots, h_N=1}^p \left(\frac{q^{kh_N}}{p^{1/2}} \right)^{e_N} \prod_{a=1}^{[N]} \alpha_a(\eta_a^{(h_a)}) \prod_{1 \leq b < a \leq [N]} \left((\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2 \right) \frac{|\eta_1^{(h_1)}, \dots, \eta_N^{(h_N)}|}{\prod_{b=1}^{[N]} \omega_b(\eta_b^{(h_b)})}, \quad (4.1)$$

$$|\beta_k\rangle = \sum_{h_1, \dots, h_N=1}^p \left(\frac{q^{-kh_N}}{p^{1/2}} \right)^{e_N} \prod_{a=1}^{[N]} \beta_a(\eta_a^{(h_a)}) \prod_{1 \leq b < a \leq [N]} \left((\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2 \right) \frac{|\eta_1^{(h_1)}, \dots, \eta_N^{(h_N)}\rangle}{\prod_{b=1}^{[N]} \omega_b(\eta_b^{(h_b)})}, \quad (4.2)$$

where the index k appears only for N even. It is easy to see that such states generate the whole space of states of the model (in particular the T-eigenbasis is just of this type). The interest toward these kind of states is due to the following:

Proposition 4.1. *Let $\langle \alpha_k |$ and $|\beta_{k'}\rangle$ two separate states of the form (4.1) and (4.2), respectively, then it holds:*

$$\langle \alpha_k | \beta_{k'} \rangle = \delta_{k,k'}^{e_N} \det_{[N]} ||\mathcal{M}_{a,b}^{(\alpha,\beta)}|| \quad \text{with} \quad \mathcal{M}_{a,b}^{(\alpha,\beta)} \equiv \left(\eta_a^{(0)} \right)^{2(b-1)} \sum_{h=1}^p \frac{\alpha_a(\eta_a^{(h)}) \beta_a(\eta_a^{(h)})}{\omega_a(\eta_a^{(h)})} q^{2(b-1)h}. \quad (4.3)$$

Proof. From the SOV-decomposition, we have:

$$\langle \alpha_k | \beta_{k'} \rangle = \left(\sum_{h_N=1}^p \frac{q^{(k-k')h_N}}{p} \right)^{e_N} \sum_{h_1, \dots, h_{[N]}=1}^p V((\eta_1^{(h_1)})^2, \dots, (\eta_{[N]}^{(h_{[N]})})^2) \prod_{a=1}^{[N]} \frac{\alpha_a(\eta_a^{(h_a)}) \beta_a(\eta_a^{(h_a)})}{\omega_a(\eta_a^{(h_a)})}, \quad (4.4)$$

where $V(x_1, \dots, x_N) \equiv \prod_{1 \leq b < a \leq [N]} (x_a - x_b)$ is the Vandermonde determinant. From this formula by using the multilinearity of the determinant w.r.t. the rows we prove the proposition. \square

In Section 2.2, we have associated to the linear space \mathcal{R}_N the structure of an Hilbert space by introducing a scalar product. Then it is clear that the above determinant formula represents also the formula for the scalar product of the two vectors $(\langle \alpha_k |)^\dagger$ and $|\beta_h\rangle$ in \mathcal{R}_N . It is worth pointing out that the vector $(\langle \alpha_k |)^\dagger \in \mathcal{R}_N$ is a separate state in the right C-eigenbasis, as it simply follows from the hermitian conjugation properties of the Yang-Baxter generators reported in Section 2.2. Then these results can be considered as the SOV analogue of the scalar product formulae [27, 142] computed in the framework of the algebraic Bethe ansatz. However, we want to stress that the determinant formulae obtained here are not restricted to the case in which one of the two states is an eigenstate of the transfer matrix, on the contrary to what happens for the scalar product formulae in the framework of the algebraic Bethe ansatz. Finally, we can prove directly from these formula the following:

Corollary 4.1. *Transfer matrix eigenstates corresponding to different eigenvalues are orthogonal states.*

Proof. Let us denote with $|t_k\rangle$ and $|t'_h\rangle$ two eigenstates of $T(\lambda)$ with eigenvalues $t_k(\lambda)$ and $t'_h(\lambda)$ for N odd and with Θ eigenvalues q^k and q^h for N even. To prove the corollary, we have to prove that:

$$\det_{[N]} \|\Phi_{a,b}^{(t,t')}\| = 0 \quad \text{with} \quad \Phi_{a,b}^{(t,t')} \equiv \left(\eta_a^{(0)}\right)^{2(b-1)} \sum_{c=1}^p \frac{Q_{t'}(\eta_a^{(c)}) \bar{Q}_t(\eta_a^{(c)})}{\omega_a(\eta_a^{(c)})} q^{2(b-1)c}, \quad (4.5)$$

with $h = k$ for N even. To prove (3.27), it is enough to show the existence of a non-zero vector $V^{(t,t')}$ such that:

$$\sum_{b=1}^{[N]} \Phi_{a,b}^{(t,t')} V_b^{(t,t')} = 0 \quad \forall a \in \{1, \dots, [N]\}. \quad (4.6)$$

For simplicity, we construct this vector in the gauge (3.27) where it results:

$$\omega_a(\eta_a^{(h)}) = \left(\eta_a^{(h)}\right)^{[N]-1}. \quad (4.7)$$

Let us recall that the eigenvalues of the transfer matrix are even Laurent polynomials of degree \bar{N} of the form:

$$t_h(\lambda) = e_N \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\pm 1}}{i} \right) (q^h + q^{-h})(\lambda^N + \lambda^{-N}) + \sum_{b=1}^{[N]} c_b \lambda^{-[N]-1+2b}, \quad (4.8)$$

$$t'_h(\lambda) = e_N \left(\prod_{a=1}^N \frac{\kappa_a \xi_a^{\pm 1}}{i} \right) (q^h + q^{-h})(\lambda^N + \lambda^{-N}) + \sum_{b=1}^{[N]} c'_b \lambda^{-[N]-1+2b}, \quad (4.9)$$

so if we define:

$$V_b^{(t,t')} \equiv c'_b - c_b \quad \forall b \in \{1, \dots, [N]\}, \quad (4.10)$$

it results:

$$\sum_{b=1}^{[N]} \Phi_{a,b}^{(t,t')} V_b^{(t,t')} = \sum_{c=1}^p Q_{t'}(\eta_a^{(c)}) \bar{Q}_t(\eta_a^{(c)}) (t'_h(\eta_a^{(c)}) - t_h(\eta_a^{(c)})). \quad (4.11)$$

We can use now the Baxter equations (2.61) and (3.34), with the chosen gauge (3.27), to rewrite:

$$\begin{aligned}
Q_{t'}(\eta_a^{(h_a)})\bar{Q}_t(\eta_a^{(h_a)})(t'(\eta_a^{(h_a)}) - t(\eta_a^{(h_a)})) &= a(\eta_a^{(h_a)})Q_{t'}(\eta_a^{(h_a-1)})\bar{Q}_t(\eta_a^{(h_a)}) \\
&+ d(\eta_a^{(h_a)})Q_{t'}(\eta_a^{(h_a+1)})\bar{Q}_t(\eta_a^{(h_a)}) \\
&- d(\eta_a^{(h_a-1)})Q_{t'}(\eta_a^{(h_a)})\bar{Q}_t(\eta_a^{(h_a-1)}) \\
&- a(\eta_a^{(h_a+1)})Q_{t'}(\eta_a^{(h_a)})\bar{Q}_t(\eta_a^{(h_a+1)}), \tag{4.12}
\end{aligned}$$

and by substituting it in (4.11) we get (4.6). \square

4.2 Decomposition of the identity in the T-eigenbasis

Let us remark that the diagonalizability and simplicity of the transfer matrix spectrum implies the following decomposition of the identity in the left and right T-eigenbasis:

$$\mathbb{I} = \sum_{k=0}^{e_N(p-1)/2} \sum_{t_k(\lambda) \in \Sigma_T^k} \frac{|t_k\rangle\langle t_k|}{\langle t_k|t_k\rangle}, \tag{4.13}$$

where

$$\langle t_k|t_k\rangle = \det_{[N]} \|\Phi_{a,b}^{(t_k,t_k)}\| \quad \text{with} \quad \Phi_{a,b}^{(t_k,t_k)} \equiv (\eta_a^{(0)})^{2(b-1)} \sum_{c=1}^p \frac{Q_t(\eta_a^{(c)})\bar{Q}_t(\eta_a^{(c)})}{\omega_a(\eta_a^{(c)})} q^{2(b-1)c}, \tag{4.14}$$

is the action of the covector $\langle t_k|$ on the vector $|t_k\rangle$ as defined in Section 3.3. It is worth to note that in the representations which defines a normal transfer matrix $T(\lambda)$, the generic covector $\langle t_k| \equiv (|t_k\rangle)^\dagger$, dual to the right T-eigenstate $|t_k\rangle$, is itself a left eigenstate of $T(\lambda)$ which taking into account the simplicity of the T-spectrum has to satisfy the following identity $\langle t_k| \equiv \alpha_{t_k} \langle t_k|$, where $\langle t_k|$ is the left T-eigenstate defined in (3.33). Of course, in these representations the following identities hold:

$$\frac{|t_k\rangle\langle t_k|}{\langle t_k|t_k\rangle} = \frac{|t_k\rangle\langle t_k|}{\| |t_k\rangle \|^2} \tag{4.15}$$

where $\| |t_k\rangle \|$ is the positive norm of the eigenvector $|t_k\rangle$ in the Hilbert space \mathcal{R}_N . The above discussion implies the relevance of computing explicitly the norm of the transfer matrix eigenstates (3.32) as it allows to fix the relative normalization α_{t_k} thanks to the identity $\alpha_{t_k} = \| |t_k\rangle \|^2 / \langle t_k|t_k\rangle$ and then it allows to take these left and right states as the one being the exact dual of the other. This interesting issue is currently under analysis.

5 SOV-representation of local operators

The determination of the scalar product formulae, presented in the previous section, is the first main step to compute matrix elements of local operators. The second one is to get the reconstruction of local operators in terms of the generators of the Yang-Baxter algebra, i.e. to invert the map which from the local operators in

the Lax matrices leads to the monodromy matrix elements. Indeed, the solution of such an inverse problem allows to compute the action of local operators on the eigenstates of the transfer matrix. Together with the scalar product formulae it leads to the determination of the matrix elements of local operators.

The first reconstruction of local operators has been achieved in [27], in the case of the XXZ spin 1/2 chain. In [29], it has been extended to fundamental lattice models, i.e. those with isomorphic auxiliary and local quantum space, for which the monodromy matrix becomes the permutation operator at a special value of the spectral parameter. The reconstruction also applies to non-fundamental lattice models, as it was shown in [29] for the higher spin XXX chains by using the fusion procedure [150]. In the case of the sine-Gordon model this type of reconstruction is still missing and the only known results are those given by T. Oota based on the use of quantum projectors [151]. However, it is worth recalling that Oota's results only lead to the reconstruction of some local operators of the lattice sine-Gordon model. In this section, we will show how to obtain all the local operators of the sine-Gordon model for the cyclic representations which occur at rational values of the coupling constant β^2 .

5.1 Oota's reconstruction of a class of local operators

Here we recall some of the results of Oota [151] which lead to the reconstruction of a certain class of local operators in the sine-Gordon model.

The Lax operator $L_n(\lambda)$ has the following factorization in terms of quantum projectors:

$$L_n(\mu_{n,+}) = P_{n,+} Q_{n,+} \equiv \kappa_n \begin{pmatrix} u_n^{1/2} (v_n \kappa_n + v_n^{-1} \kappa_n^{-1}) \\ u_n^{-1/2} (v_n \kappa_n^{-1} + v_n^{-1} \kappa_n) \end{pmatrix} \begin{pmatrix} u_n^{1/2} & u_n^{-1/2} \end{pmatrix}, \quad (5.1)$$

$$L_n(\mu_{n,-}) = P_{n,-} Q_{n,-} \equiv \kappa_n \begin{pmatrix} u_n^{1/2} \\ u_n^{-1/2} \end{pmatrix} \begin{pmatrix} (v_n \kappa_n + v_n^{-1} \kappa_n^{-1}) u_n^{1/2} & (v_n \kappa_n^{-1} + v_n^{-1} \kappa_n) u_n^{-1/2} \end{pmatrix}, \quad (5.2)$$

when computed in the zeros $\mu_{n,\pm}$ of the quantum determinant; such factorization properties are at the basis of the following Oota's reconstruction.

Proposition 5.1. *The local operators u_n and $\alpha_{0,n} \equiv ((q^{-1}v_n^2 + \kappa_n^2)/(q^{-1}v_n^2 \kappa_n^2 + 1)) u_n^{-1}$ admit the reconstructions:*

$$u_n = U_n B^{-1}(\mu_{n,+}) A(\mu_{n,+}) U_n^{-1} = U_n D^{-1}(\mu_{n,+}) C(\mu_{n,+}) U_n^{-1}, \quad (5.3)$$

$$\alpha_{0,n} = U_n A^{-1}(\mu_{n,-}) B(\mu_{n,-}) U_n^{-1} = U_n C^{-1}(\mu_{n,-}) D(\mu_{n,-}) U_n^{-1}, \quad (5.4)$$

where the shift operator U_n brings the quantum sites from 1 to $n-1$ to the right end of the chain:

$$U_n M_{1,\dots,N}(\lambda) U_n^{-1} \equiv M_{n,\dots,N,1,\dots,n-1}(\lambda) \equiv L_{n-1}(\lambda) \cdots L_1(\lambda) L_N(\lambda) \cdots L_n(\lambda). \quad (5.5)$$

It is then clear that the formulae (5.3)-(5.4) allow to reconstruct all the powers $u_n^k = U_n (B^{-1}(\mu_{n,+}) A(\mu_{n,+}))^k U_n^{-1}$ of the local operators u_n but they do not give a direct reconstruction of the local operators v_n ; indeed, only rational functions like $(q^{-1}v_n^2 + \kappa_n^2)/(q^{-1}v_n^2 \kappa_n^2 + 1)$ are obtained.

Let us make some comments on the shift operators U_n . The definition (5.5) characterizes the shift operator U_n up to a constant and implies, by the cyclicity invariance of the trace, their commutativity with the transfer matrix $T(\lambda)$. Moreover, in the case of even chain, the shift operators U_n clearly commute also with the Θ -charge. Then, in the cyclic representations of the sine-Gordon model under consideration, the simplicity of the transfer matrix spectrum implies:

$$U_n |t_k\rangle = \varphi_n^{(t_k)} |t_k\rangle, \quad (5.6)$$

where $|t_k\rangle$ is the generic eigenstate of $T(\lambda)$ for odd chain and of $(T(\lambda), \Theta)$ for the even chain. In particular, this implies that the shift operators only produce a prefactor in the form factors of local operators which is one if left and right eigenstates are dual of each other¹². It is worth remarking that in¹³ [132], for the special case of highest weight representations of the even lattice sine-Gordon model, it has been shown that:

$$U_n |t\rangle \propto \prod_{a=1}^{n-1} t(\mu_a) |t\rangle \propto \prod_{a=1}^{n-1} \frac{Q_t(\mu_a/q)}{Q_t(\mu_a)} |t\rangle, \quad (5.7)$$

where μ_a are zeros of the quantum determinant in these representations. This result is interesting as it shows that the shift operators U_n for non-fundamental lattice models, like the sine-Gordon model, are characterized by the same type of eigenvalues they have in fundamental lattice models, like the XXZ spin 1/2 chain. However, the proof of (5.7) presented in [132] is representation dependent, as it is based on the algebraic Bethe ansatz representation of the transfer matrix eigenstates. Then, an independent proof is required for cyclic representations of the sine-Gordon model and it will be given directly in a future publication [152] in the more general cyclic representations of the 6-vertex Yang-Baxter algebra associated to the τ_2 -model.

5.2 Inverse problem solution for all local operators

Here, we show how to complete the reconstruction of local operators by solving the inverse problem for the local operators v_n and their powers. The main ingredient used will be the cyclicity of the representations of the sine-Gordon model here analyzed.

Let us define the local operators:

$$\beta_{k,n} \equiv u_n^k \alpha_{0,n} u_n^{1-k} = \frac{q^{2k-1} v_n^2 + \kappa_n^2}{q^{2k-1} v_n^2 \kappa_n^2 + 1}, \quad (5.8)$$

then the following proposition holds:

Proposition 5.2. *In the cyclic representations of the sine-Gordon model, the local operators v_n^{2k} admit the following reconstruction:*

$$v_n^{2k} = \frac{(-1)^k (v_n^{2p} \kappa_n^{2p} + 1)}{p \kappa_n^{2k} (\kappa_n^2 - \kappa_n^{-2})} \sum_{a=0}^{p-1} q^{-k(2a-1)} \beta_{a,n}, \quad \text{for } k \in \{1, \dots, p-1\}, \quad (5.9)$$

where:

$$\beta_{k,n} = U_n \left[(B^{-1}(\mu_{n,+}) A(\mu_{n,+}))^k A^{-1}(\mu_{n,-}) B(\mu_{n,-}) (B^{-1}(\mu_{n,+}) A(\mu_{n,+}))^{1-k} \right] U_n^{-1}. \quad (5.10)$$

¹²Translational invariance for the limit of the homogeneous chain.

¹³This result is there attributed to V. Korepin, private communications.

Proof. In our cyclic representations the local operators u_n and v_n satisfy the property that u_n^p and v_n^p are central, i.e. u_n^p and v_n^p are just numbers u_n^p and v_n^p which characterize our representations. Then the following identity holds:

$$\prod_{j=0}^{p-1} (q^{2j-1} v_n^2 \kappa_n^2 + 1) = 1 + v_n^{2p} \kappa_n^{2p}, \quad (5.11)$$

and so:

$$\frac{v_n^{2p} \kappa_n^{2p} + 1}{q^{2k-1} v_n^2 \kappa_n^2 + 1} = \sum_{a=0}^{p-1} (-1)^a q^{a(2k-1)} v_n^{2a} \kappa_n^{2a}. \quad (5.12)$$

The previous formula allows to rewrite the rational function $\beta_{k,n}$ as a finite sum in powers of v_n^2 :

$$\beta_{k,n} = \frac{v_n^{2p} \kappa_n^{2(p-1)} + \kappa_n^2 + (\kappa_n^2 - \kappa_n^{-2}) \sum_{a=1}^{p-1} (-1)^a q^{a(2k-1)} v_n^{2a} \kappa_n^{2a}}{v_n^{2p} \kappa_n^{2p} + 1}, \quad (5.13)$$

then by taking the discrete Fourier transformation, we get the reconstructions (5.9), plus the sum rules:

$$\sum_{a=0}^{p-1} \beta_{a,n} = \frac{p v_n^{2p} \kappa_n^{2(p-1)} + \kappa_n^2}{v_n^{2p} \kappa_n^{2p} + 1}. \quad (5.14)$$

Finally, the representation (5.10) for the $\beta_{a,n}$ are trivially derived by (5.3)-(5.4). \square

Note that thanks to the identities $v_n^k = v_n^{2h} / v_n^p$ for $k = 2h - p$ odd integer smaller than p , the formulae (5.9) indeed lead to the reconstruction of all the powers v_n^k for $k \in \{1, \dots, p-1\}$. Then the previous proposition together with Oota's reconstructions leads to the announced complete reconstruction of local operators for cyclic representations of the sine-Gordon model.

5.3 SOV-representations of all local operators

In order to compute the action of the local operators v_n^k and u_n^k on eigenstates of the transfer matrix and eventually obtain their form factors, it is necessary to first derive their SOV-representations¹⁴. Here, we show how these SOV-representations can be obtained from the previously given solutions of the inverse problem. In order to simplify the presentation, we introduce here explicitly the operators¹⁵ $\eta_1, \dots, \eta_N, \eta_A$ and η_D defined by the following actions on the generic element $\langle \mathbf{k} | \equiv \langle \eta_1^{(k_1)}, \dots, \eta_N^{(k_N)} |$ of the left B-eigenbasis:

$$\langle \mathbf{k} | \eta_a = \eta_a^{(k_a)} \langle \mathbf{k} |, \quad \forall a \in \{1, \dots, N\}, \quad \langle \mathbf{k} | \eta_A = \eta_{\mathbf{k},A} \langle \mathbf{k} | \quad \text{and} \quad \langle \mathbf{k} | \eta_D = \eta_{\mathbf{k},D} \langle \mathbf{k} |, \quad (5.15)$$

where the corresponding eigenvalues $\eta_a^{(k_a)}$, $\eta_{\mathbf{k},A}$ and $\eta_{\mathbf{k},D}$ are the complex numbers defined in Section 2.3. The following lemma is important as it solves the combinatorial problem related to the computations of the SOV-representations of monomials in the Yang-Baxter generators:

¹⁴To simplify the notations we chose to present the results in this subsection only for the case N odd and for the representations with $v_n^p = u_n^p = 1$.

¹⁵To simplify the exposition, we have decided to keep as simple as possible the notations for these operators, we hope that nevertheless the difference with the corresponding eigenvalues is clear.

Lemma 5.1. *The operator $(B^{-1}(\lambda)A(\lambda))^k$ has the following left SOV-representation:*

$$\begin{aligned} (B^{-1}(\lambda)A(\lambda))^k &= K^{-k} \sum_{\substack{\bar{\alpha} \equiv \{\alpha_1, \dots, \alpha_N\} \in \mathbb{N}_0^N: \\ \sum_{h=1}^N \alpha_h = k}} \left[\begin{matrix} k \\ \bar{\alpha} \end{matrix} \right] \prod_{j=1}^N \left(\prod_{h=0}^{\alpha_j-1} \frac{a(\eta_j q^{-h})}{(\lambda q^h / \eta_j - \eta_j / \lambda q^h)} \right) \\ &\quad \times \prod_{i \neq j, i=1}^N \prod_{h=\alpha_i-\alpha_j+1}^{\alpha_i} \frac{1}{(\eta_j q^h / \eta_i - \eta_i / \eta_j q^h)} \prod_{j=1}^N T_j^{-\alpha_j}, \end{aligned} \quad (5.16)$$

acting on the state $\langle \eta_1, \dots, \eta_N |$, where:

$$\left[\begin{matrix} k \\ \bar{\alpha} \end{matrix} \right] \equiv \frac{[k]!}{\prod_{j=1}^N [\alpha_j]!}, \quad [k]! \equiv [k][k-1] \cdots [1], \quad [a] \equiv \frac{q^a - q^{-a}}{q - q^{-1}}. \quad (5.17)$$

Proof. Here we use the commutation relations:

$$T_a^\pm \eta_b = q^{\pm \delta_{ab}} \eta_b T_a^\pm, \quad (5.18)$$

and the following left SOV-representation:

$$B^{-1}(\lambda)A(\lambda) = K^{-1} \sum_{a=1}^N \frac{a(\eta_a)}{(\lambda/\eta_a - \eta_a/\lambda)} \prod_{b \neq a} \frac{1}{(\eta_a/\eta_b - \eta_b/\eta_a)} T_a^{-1}, \quad (5.19)$$

then the lemma holds for $k = 1$ and we prove it by induction for $k > 1$. Let us take N integers α_i :

$$\sum_{a=1}^N \alpha_i = k, \quad (5.20)$$

from which we define the set of integers $I = \{i \in \{1, \dots, N\} : \alpha_i \neq 0\}$ and $C_{\bar{\alpha}}^{(k)}$ as the coefficient of $\prod T_i^{-\alpha_i}$ in the expansion of the k -th power of $B^{-1}(\lambda)A(\lambda)$. By writing $(B^{-1}(\lambda)A(\lambda))^k = (B^{-1}(\lambda)A(\lambda))^{k-1} B^{-1}(\lambda)A(\lambda)$ and by using the induction hypothesis for the power $k-1$ of $B^{-1}(\lambda)A(\lambda)$, we have:

$$\begin{aligned} C_{\bar{\alpha}}^{(k)} &= K^{-k} \sum_{a \in I} \left[\begin{matrix} k-1 \\ \bar{\alpha} - \bar{\delta}_a \end{matrix} \right] \\ &\quad \prod_{j=1}^N \prod_{h=0}^{\alpha_j - \delta_{a,j} - 1} \left(\frac{a(\eta_j q^{-h})}{(\lambda q^h / \eta_j - \eta_j / \lambda q^h)} \times \prod_{i \neq j, i=1}^N \frac{1}{q^{\alpha_i - \delta_{a,i} - h} \eta_j / \eta_i - \eta_i / q^{\alpha_i - \delta_{a,i} - h} \eta_j} \right) \\ &\quad \times \frac{a(\eta_a q^{-\alpha_a + 1})}{(\lambda q^{\alpha_a - 1} / \eta_a - \eta_a / \lambda q^{\alpha_a - 1})} \prod_{i \in I \setminus \{a\}} \frac{1}{q^{\alpha_a - \alpha_i - 1} \eta_i / \eta_a - \eta_a / q^{\alpha_a - \alpha_i - 1} \eta_i}, \end{aligned} \quad (5.21)$$

with $\bar{\delta}_a \equiv (\delta_{1,a}, \dots, \delta_{N,a})$. The first term in the r.h.s. is the coefficient of $\prod T_i^{-\alpha_i + \delta_{a,i}}$ in $(B^{-1}(\lambda)A(\lambda))^{k-1}$ and the second is the coefficient of T_a^{-1} in $B^{-1}(\lambda)A(\lambda)$ once the commutations between $\prod T_i^{-\alpha_i + \delta_{a,i}}$ and

the η_i have been performed. This can be rewritten as:

$$C_{\bar{\alpha}}^{(k)} = \kappa^{-k} \frac{[k-1]!}{\prod [\alpha_i]!} \left(\prod_{j=1}^N \prod_{h=0}^{\alpha_j-1} \left(\prod_{i \neq j, i=1}^N \frac{1}{q^{\alpha_i-h} \eta_j / \eta_i - \eta_i / q^{\alpha_i-h} \eta_j} \right) \frac{a(q^{-h} \eta_j)}{(\lambda q^h / \eta_j - \eta_j / \lambda q^h)} \right) \\ \times \sum_{a \in I} ([\alpha_a] \prod_{i \in I \setminus \{a\}} \frac{q^{\alpha_a} \eta_i / \eta_a - \eta_a / q^{\alpha_a} \eta_i}{q^{\alpha_a - \alpha_i} \eta_i / \eta_a - \eta_a / q^{\alpha_a - \alpha_i} \eta_i}), \quad (5.22)$$

which leads to our result when we use the relation:

$$\sum_{a=1}^n [\alpha_a] \prod_{i \neq a} \frac{q^{\alpha_a} \eta_i / \eta_a - \eta_a / q^{\alpha_a} \eta_i}{q^{\alpha_a - \alpha_i} \eta_i / \eta_a - \eta_a / q^{\alpha_a - \alpha_i} \eta_i} = \left[\sum_{a=1}^n \alpha_a \right]. \quad (5.23)$$

The above formula holds for any n , for any set of numbers η_i and for any non-negative integers α_i , all of them being in generic position. This is shown by studying the contour integral and the residues of the function:

$$g(z) = \frac{1}{z} \prod_{i=1}^n \frac{z - \eta_i^2}{z - q^{-2\alpha_i} \eta_i^2}. \quad (5.24)$$

□

Remark 5. Let us point out that the power p of $B^{-1}(\lambda)A(\lambda)$ is a central element of the Yang-Baxter algebra and it reads:

$$(B^{-1}(\lambda)A(\lambda))^p = \mathcal{B}(\Lambda)^{-1} \mathcal{A}(\Lambda), \quad (5.25)$$

as it simply follows from the commutations relations:

$$B^{-1}(q\lambda)A(q\lambda) = A(\lambda)B^{-1}(\lambda). \quad (5.26)$$

Then, it is important to verify that the same result follows from the previous lemma for $k = p$. In order to prove it, it is enough to use the following properties of the quantum binomials:

$$\begin{bmatrix} p \\ \bar{\alpha} \end{bmatrix} = \begin{cases} 1 & \text{if } \exists i \in \{1, \dots, N\} : \alpha_i = p\delta_{a,i} \forall a \in \{1, \dots, N\}, \\ 0 & \text{otherwise,} \end{cases} \quad (5.27)$$

from which the formula (5.16) for $k = p$ reads:

$$(B^{-1}(\lambda)A(\lambda))^p = \kappa^{-p} \sum_{a=1}^p \frac{\prod_{k=1}^p a(q^k \eta_a)}{(\Lambda/Z_a - Z_a/\Lambda)} \prod_{b \neq a} \frac{1}{(Z_b/Z_a - Z_a/Z_b)}, \quad (5.28)$$

and to observe that the r.h.s. of (5.28) indeed coincides with $\mathcal{B}(\Lambda)^{-1} \mathcal{A}(\Lambda)$.

Note that the above lemma gives directly the SOV-representations of the local operators u_n^k and $\alpha_{0,n}^{-k}$ with $k \in \{1, \dots, p-1\}$ when we fix the parameter λ to $\mu_{n,\varepsilon}$ with $\varepsilon = +$ and $\varepsilon = -$, respectively. Moreover, it allows to derive also the SOV-representations of the local operators v_n^k as it follows:

Corollary 5.1. *The local operators v_n^{2k} with $k \in \{1, \dots, p-1\}$ have the following left SOV-representation:*

$$\begin{aligned} U_n^{-1} v_n^{2k} U_n &= v_n^{(2k)} + \sum_{a=1}^N \sum_{\substack{\bar{\alpha} \equiv \{\alpha_1, \dots, \alpha_N\} \in \mathbb{N}_0^N: \\ \sum_{h=1}^N \alpha_h = p-1}} \left[\begin{matrix} p-1 \\ \bar{\alpha} \end{matrix} \right] \prod_{j=1}^N \left(\prod_{h=0}^{\alpha_j-1} \frac{a(\eta_j q^{-h})}{(\mu_{n,-} q^h / \eta_j - \eta_j / \mu_{n,-} q^h)} \right) \\ &\quad \times \prod_{i \neq j, i=1}^N \prod_{h=\alpha_i - (\alpha_j + \delta_{j,a}) + 1}^{\alpha_i} \frac{1}{(\eta_j q^h / \eta_i - \eta_i / \eta_j q^h)} \Bigg) v_{n,(a,\bar{\alpha})}^{(2k)} \prod_{j=1}^N T_j^{-(\alpha_j + \delta_{j,a})}, \end{aligned} \quad (5.29)$$

where:

$$v_n^{(2k)} = \frac{(-1)^k (\kappa_n^{2p} + 1)}{p \kappa_n^{2k} (\kappa_n^2 - \kappa_n^{-2})} \sum_{r=1}^p \frac{q^{-k(2r-1)} (q^r - q^{-r})}{q^r \kappa_n^2 - q^{-r} \kappa_n^{-2}}, \quad (5.30)$$

$$\begin{aligned} v_{n,(a,\bar{\alpha})}^{(2k)} &\equiv \frac{(-1)^k (\kappa_n^{2p} + 1)}{p \kappa_n^{2k} (\kappa_n^2 - \kappa_n^{-2})} \sum_{r=1}^p \frac{q^{-k(2r-1)} (\kappa_n^2 - \kappa_n^{-2}) a(\eta_a q^{-\alpha_a})}{(q^r \kappa_n^2 - q^{-r} \kappa_n^{-2}) (\mu_{n,+} q^{\alpha_a+r} / \eta_a - \eta_a / \mu_{n,+} q^{\alpha_a+r})} \\ &\quad \times \prod_{j=1}^N \prod_{h=0}^{r-1} \frac{(\mu_{n,+} q^{\alpha_j+h} / \eta_j - \eta_j / \mu_{n,+} q^{\alpha_j+h})}{(\mu_{n,+} q^h / \eta_j - \eta_j / \mu_{n,+} q^h)}. \end{aligned} \quad (5.31)$$

Proof. This is a consequence of the previous lemma and of the identities:

$$U_n^{-1} \beta_{k,n} U_n = \frac{\kappa_n^2 - \kappa_n^{-2}}{q^k \kappa_n^2 - q^{-k} \kappa_n^{-2}} \gamma_{k,n} + \frac{q^k - q^{-k}}{q^k \kappa_n^2 - q^{-k} \kappa_n^{-2}}, \quad \text{for } k \in \{1, \dots, p-1\}, \quad (5.32)$$

where:

$$\gamma_{k,n} = B^{-1}(\mu_{n,+}^p) \prod_{j=k}^{p-1} B(\mu_{n,+} q^j) A^{-1}(\mu_{n,-}) B(\mu_{n,-}) B^{-1}(\mu_{n,+} q^k) A(\mu_{n,+} q^k) \prod_{j=0}^{k-1} B(\mu_{n,+} q^j), \quad (5.33)$$

Now by using the relations:

$$A^{-1}(\mu_{n,-}) B(\mu_{n,-}) = (B^{-1}(\mu_{n,-}) A(\mu_{n,-}))^{p-1}, \quad (5.34)$$

we get the following representations:

$$\begin{aligned} \gamma_{k,n} &= \sum_{a=1}^N \sum_{\substack{\bar{\alpha} \equiv \{\alpha_1, \dots, \alpha_N\} \in \mathbb{N}_0^N: \\ \sum_{h=1}^N \alpha_h = p-1}} \left[\begin{matrix} p-1 \\ \bar{\alpha} \end{matrix} \right] \prod_{j=1}^N \left(\prod_{h=0}^{\alpha_j-1} \frac{a(\eta_j q^{-h})}{(\mu_{n,-} q^h / \eta_j - \eta_j / \mu_{n,-} q^h)} \right) \\ &\quad \times \prod_{i \neq j, i=1}^N \prod_{h=\alpha_i - (\alpha_j + \delta_{j,a}) + 1}^{\alpha_i} \frac{1}{(\eta_j q^h / \eta_i - \eta_i / \eta_j q^h)} \Bigg) \frac{a(\eta_a q^{-\alpha_a})}{(\mu_{n,+} q^{\alpha_a+r} / \eta_a - \eta_a / \mu_{n,+} q^{\alpha_a+r})} \\ &\quad \times \prod_{j=1}^N \prod_{h=0}^{r-1} \frac{(\mu_{n,+} q^{\alpha_j+h} / \eta_j - \eta_j / \mu_{n,+} q^{\alpha_j+h})}{(\mu_{n,+} q^h / \eta_j - \eta_j / \mu_{n,+} q^h)} \prod_{j=1}^N T_j^{-(\alpha_j + \delta_{j,a})}, \end{aligned} \quad (5.35)$$

from which our result follows. \square

6 Form factors of local operators

In the following we will compute matrix elements (form factors) of the form¹⁶:

$$\langle t | O_n | t' \rangle \quad (6.1)$$

which by definition are the action of a covector $\langle t | \in \mathcal{L}_N$, a left T -eigenstate defined in (3.33), on the vector obtained by the action of the local operator O_n on the right T -eigenstate $|t'\rangle \in \mathcal{R}_N$ defined in (3.32). Of course, these form factors depend on the normalization of the states $\langle t |$ and $|t'\rangle$ and then it is worth pointing out that nevertheless we can use them to expand m-point functions like:

$$\frac{\langle t | O_{n_1} \cdots O_{n_m} | t \rangle}{\langle t | t \rangle}. \quad (6.2)$$

Indeed, by definition these m-point functions are normalization independent and by using m-1 times the decomposition of the identity (4.13), we get the expansions:

$$\frac{\langle t | O_{n_1} \cdots O_{n_m} | t \rangle}{\langle t | t \rangle} = \sum_{t^{(1)}(\lambda), \dots, t^{(m-1)}(\lambda) \in \Sigma_T} \frac{\langle t | O_{n_1} | t^{(1)} \rangle \langle t^{(m-1)} | O_{n_m} | t \rangle \prod_{a=2}^{m-1} \langle t^{(a-1)} | O_{n_a} | t^{(a)} \rangle}{\langle t | t \rangle \prod_{a=1}^{m-1} \langle t^{(a)} | t^{(a)} \rangle}, \quad (6.3)$$

where in the r.h.s there are exactly the form factors (6.1) that we are going to compute in this paper.

6.1 Form factors of u_n

In this section we use the SOV-representations of local operators to give some examples of completely resummed form factors.

Proposition 6.1. *Let $\langle t_k |$ and $|t'_{k'}\rangle$ be two eigenstates of the transfer matrix $T(\lambda)$, then it holds:*

$$\langle t_k | u_n | t'_{k'} \rangle = \frac{\varphi_n^{(t_k)}}{\varphi_n^{(t'_{k'})}} (\delta_{k, k'+1})^{\mathbf{e}_N} \det_{[N]} (||\mathcal{U}_{a,b}^{(t,t')}(\mu_{n,+})||), \quad (6.4)$$

where $\varphi_n^{(t_k)}$ and $\varphi_n^{(t'_{k'})}$ are the eigenvalues of the shift operator U_n and $||\mathcal{U}_{a,b}^{(t,t')}(\lambda)||$ is the $[N] \times [N]$ matrix:

$$\mathcal{U}_{a,b}^{(t,t')}(\lambda) \equiv \Phi_{a,b+1/2}^{(t,t')} \text{ for } b \in \{1, \dots, [N] - 1\}, \quad (6.5)$$

$$\begin{aligned} \mathcal{U}_{a,[N]}^{(t,t')}(\lambda) \equiv & \frac{(\eta_a^{(0)})^{[N]-1}}{K \eta_N^{(0) \mathbf{e}_N}} \sum_{h=1}^p \frac{q^{([N]-1)h} Q_{t'}(\eta_a^{(h)})}{\omega_a(\eta_a^{(h)})} \left[\frac{\bar{Q}_t(\eta_a^{(h+1)})}{(\lambda/\eta_a^{(h+1)} - \eta_a^{(h+1)}/\lambda)} \bar{a}(\eta_a^{(h)}) \right. \\ & \left. + \mathbf{e}_N \bar{Q}_t(\eta_a^{(h)}) \left(\frac{\lambda}{\prod_{j=1}^N \xi_j} (\eta_a^{(h)})^{[N]} q^{k'} + \frac{\prod_{j=1}^N \xi_j}{\lambda} (\eta_a^{(h)})^{-[N]} q^{-k'} \right) \right], \end{aligned} \quad (6.6)$$

where $K \equiv \prod_{n=1}^N \kappa_n / i$.

¹⁶To simplify the notations in this introduction to Section 6 we are omitting the index $k \in \{0, \dots, (p-1)/2\}$ in the transfer matrix eigenstates which are required in the case of even chain.

Proof. The right SOV-representation of the operator $B^{-1}(\lambda)A(\lambda)$ reads:

$$B^{-1}(\lambda)A(\lambda)|\mathbf{k}\rangle = \frac{e_N}{\eta_N} \left(\frac{\lambda}{\eta_A} T_N^+ + \frac{\eta_A}{\lambda} T_N^- \right) |\mathbf{k}\rangle + \sum_{a=1}^{[N]} T_a^+ |\mathbf{k}\rangle \frac{\bar{a}(\eta_a)}{K \bar{\eta}_N^{e_N} (\lambda/\eta_a q - \eta_a q/\lambda)} \prod_{b \neq a} \frac{1}{(\eta_a/\eta_b - \eta_b/\eta_a)}, \quad (6.7)$$

Let us denote with $[B^{-1}(\lambda)A(\lambda)]$ the second term on the r.h.s. of (6.7). Then, we observe that from the SOV-decomposition of the T -eigenstates, we have:

$$\begin{aligned} \langle t_k | [B^{-1}(\lambda)A(\lambda)] | t'_{k'} \rangle &= \left(\frac{\sum_{h_N=1}^p q^{(k-1-k')h_N}}{p \eta_N^{(0)}} \right)^{e_N} \sum_{a=1}^{[N]} \sum_{h_1, \dots, h_{[N]}=1}^p V((\eta_1^{(h_1)})^2, \dots, (\eta_{[N]}^{(h_{[N]})})^2) \\ &\times \prod_{b \neq a, b=1}^{[N]} \frac{\eta_b^{(h_b)} Q_{t'}(\eta_b^{(h_b)}) \bar{Q}_t(\eta_b^{(h_b)})}{\omega_b(\eta_b^{(h_b)}) ((\eta_a^{(h_a)})^2 - (\eta_b^{(h_b)})^2)} \\ &\times \frac{\bar{Q}_t(\eta_a^{(h_a+1)}) Q_{t'}(\eta_a^{(h_a)})}{\omega_a(\eta_a^{(h_a)}) K} \frac{(\eta_a^{(h_a)})^{[N]-1} \bar{a}(\eta_a^{(h_a)})}{(\lambda/\eta_a^{(h_a+1)} - \eta_a^{(h_a+1)}/\lambda)}, \end{aligned} \quad (6.8)$$

and so:

$$\begin{aligned} \langle t_k | [B^{-1}(\lambda)A(\lambda)] | t'_{k'} \rangle &= \left(\frac{\delta_{k,k'+1}}{\eta_N^{(0)}} \right)^{e_N} \sum_{a=1}^{[N]} \underbrace{\sum_{h_1, \dots, h_{[N]}=1}^p}_{\substack{h_a \text{ is missing.} \\ \text{(We have removed the row } a.)}} \hat{V}_a((\eta_1^{(h_1)})^2, \dots, (\eta_{[N]}^{(h_{[N]})})^2) \\ &\times \prod_{b \neq a, b=1}^{[N]} \frac{\eta_b^{(h_b)} Q_{t'}(\eta_b^{(h_b)}) \bar{Q}_t(\eta_b^{(h_b)})}{\omega_b(\eta_b^{(h_b)})} \\ &\times (-1)^{[N]+a} \sum_{h_a=1}^p \frac{\bar{Q}_t(\eta_a^{(h_a+1)}) Q_{t'}(\eta_a^{(h_a)}) (\eta_a^{(h_a)})^{[N]-1} \bar{a}(\eta_a^{(h_a)})}{\omega_a(\eta_a^{(h_a)}) K (\lambda/\eta_a^{(h_a+1)} - \eta_a^{(h_a+1)}/\lambda)}, \end{aligned} \quad (6.9)$$

bringing the sum over $(h_1, \dots, \widehat{h_a}, \dots, h_{[N]})$ inside the Vandermonde determinant \hat{V}_a , we have that the above expression is just the expansion of the determinant:

$$\langle t_k | [B^{-1}(\lambda)A(\lambda)] | t'_{k'} \rangle = (\delta_{k,k'+1})^{e_N} \det_{[N]} (|| \mathcal{U}_{a,b}^{(t,t')}(\lambda) ||), \quad (6.10)$$

where $[\mathcal{U}_{a,b}^{(t,t')}(\lambda)]$ coincides with $\Phi_{a,b+1/2}^{(t,t')}$ for $b \in \{1, \dots, [N] - 1\}$, while:

$$[\mathcal{U}_{a,[N]}^{(t,t')}(\lambda)] \equiv \frac{(\eta_a^{(0)})^{[N]-1}}{K \bar{\eta}_N^{e_N}} \sum_{h=1}^p \frac{q^{([N]-1)h} Q_{t'}(\eta_a^{(h)}) \bar{Q}_t(\eta_a^{(h+1)})}{\omega_a(\eta_a^{(h)}) (\lambda/\eta_a^{(h+1)} - \eta_a^{(h+1)}/\lambda)} \bar{a}(\eta_a^{(h)}). \quad (6.11)$$

Now let us compute the matrix elements:

$$\begin{aligned} \langle t_k | \eta_N^{-1} \eta_A^\mp T_N^\pm | t'_{k'} \rangle &= \left(\frac{\sum_{h_N=1}^p q^{(k-1-k')h_N}}{p \eta_N^{(0)} \prod_{j=1}^N \xi_j^{\pm 1}} \right)^{e_N} \sum_{h_1, \dots, h_{[N]}=1}^p V((\eta_1^{(h_1)})^2, \dots, (\eta_{[N]}^{(h_{[N]})})^2) \\ &\times \prod_{b=1}^{[N]} \frac{(\eta_b^{(h_b+k')})^{\pm 1} Q_{t'}(\eta_b^{(h_b)}) \bar{Q}_t(\eta_b^{(h_b)})}{\omega_b(\eta_b^{(h_b)})}, \end{aligned} \quad (6.12)$$

and so:

$$\langle t_k | \eta_N^{-1} \eta_A^\mp T_N^\pm | t'_{k'} \rangle = \left(\frac{q^{\pm k'} \delta_{k,k'+1}}{\eta_N^{(0)} \prod_{j=1}^N \xi_j^{\pm 1}} \right)^{e_N} \det_{[N]}(|\Phi_{a,b \pm 1/2}^{(t,t')}|). \quad (6.13)$$

By using the fact that $[N] - 1$ columns are common in the matrix of formula (6.10) and in those of (6.13), we get our result. Let us remark that the above result holds for any value of λ . \square

Remark 6. It is worth pointing out that the form factors of u_n are written in terms of a determinant of a matrix whose elements coincide with those of the scalar product, except for the last line which is modified by the presence of the local operator. It is then interesting to recall that a similar statement holds for the form factors of the local operators in the XXZ spin 1/2 chain.

6.2 Suitable operator basis for form factor computations

In this section we introduce an operator basis which can be conveniently used to describe all local operators. The interest toward this basis is due to the fact that the form factors of its elements are simple being represented by a determinant formula.

6.2.1 Basis of elementary operators

Let us introduce the following operators:

$$\mathcal{O}_{a,k} \equiv \frac{B(\eta_a^{(p+k-1)}) B(\eta_a^{(p+k-2)}) \dots B(\eta_a^{(k+1)}) A(\eta_a^{(k)})}{p \eta_N^{e_N(p-1)} K^{(p-1)} \prod_{b \neq a, b=1}^{[N]} (Z_a/Z_b - Z_b/Z_a)} \quad \text{with } k \in \{0, \dots, p-1\}, \quad (6.14)$$

where the $\eta_a^{(k)}$ are fixed in Section 2.3.

Lemma 6.1. *The operators $\mathcal{O}_{a,k}$ satisfy the following properties:*

$$\mathcal{O}_{a,k} \mathcal{O}_{a,h} \text{ is non-zero if and only if } h = k - 1, \quad (6.15)$$

and

$$\mathcal{O}_{a,k} \mathcal{O}_{a,k-1} \dots \mathcal{O}_{a,k+1-p} \mathcal{O}_{a,k-p} = \frac{\mathcal{A}(Z_a)}{\prod_{b \neq a, b=1}^{[N]} (Z_a/Z_b - Z_b/Z_a)} \mathcal{O}_{a,k}. \quad (6.16)$$

Moreover the following commutation relations hold:

$$\eta_A \mathcal{O}_{a,k} = q \mathcal{O}_{a,k} \eta_A, \quad [\eta_N, \mathcal{O}_{a,k}] = [\Theta, \mathcal{O}_{a,k}] = 0, \quad (6.17)$$

and

$$\mathcal{O}_{a,k}\mathcal{O}_{b,h} = \frac{(\eta_a^{(k-h+1)}/\eta_b^{(0)} - \eta_b^{(0)}/\eta_a^{(k-h+1)})}{(\eta_a^{(k-h-1)}/\eta_b^{(0)} - \eta_b^{(0)}/\eta_a^{(k-h-1)})} \mathcal{O}_{b,h}\mathcal{O}_{a,k} \quad (6.18)$$

for $a \neq b \in \{1, \dots, [N]\}$.

Proof. The first property follows from $\mathcal{B}(Z_a) = 0$, where $\mathcal{B}(\Lambda)$ is the average value of $B(\lambda)$. By the definition of $\mathcal{O}_{a,k}$ it is clear that:

$$\langle \eta_1^{(k_1)}, \dots, \eta_a^{(h)}, \dots, \eta_N^{(k_N)} | \mathcal{O}_{a,k} = \frac{a(\eta_a^{(k)})\delta_{h,k}}{\prod_{b \neq a, b=1}^N (\eta_a^{(k)}/\eta_b^{(k_b)} - \eta_b^{(k_b)}/\eta_a^{(k)})} \langle \eta_1^{(k_1)}, \dots, \eta_a^{(k-1)}, \dots, \eta_N^{(k_N)} |, \quad (6.19)$$

so that the second property simply follows. To prove the last property we have to use the Yang-Baxter commutation relation:

$$(\lambda/\mu - \mu/\lambda)A(\lambda)B(\mu) = (\lambda/q\mu - \mu q/\lambda)B(\mu)A(\lambda) + (q - q^{-1})B(\lambda)A(\mu) \quad (6.20)$$

we have before to move the $A(\eta_a^{(k)})$ to the right through all the $B(\eta_a^{(j)})$, remarking that only the first term of the r.h.s of (6.20) survives, and after to move the $A(\eta_a^{(h)})$ to the left. \square

Let us introduce now the following monomials which we will call *elementary operators*:

$$\mathcal{E}_{(k,k_0)^{\mathbf{e}_N}, (a_1, k_1), \dots, (a_r, k_r)}^{(\alpha_1, \dots, \alpha_r)} \equiv \eta_N^{-\mathbf{e}_N k} \left(\frac{\Theta}{\eta_A} \right)^{\mathbf{e}_N k_0} \mathcal{O}_{a_1, k_1}^{(\alpha_1)} \dots \mathcal{O}_{a_r, k_r}^{(\alpha_r)}, \quad (6.21)$$

where $\sum_{h=1}^r \alpha_h \leq p$, $k, k_i \in \{0, \dots, p-1\}$, $a_i < a_j \in \{1, \dots, [N]\}$ for $i < j \in \{1, \dots, [N]\}$ and:

$$\mathcal{O}_{a,k}^{(\alpha)} \equiv \mathcal{O}_{a,k} \mathcal{O}_{a,k-1} \dots \mathcal{O}_{a,k+1-\alpha}, \text{ with } \alpha \in \{1, \dots, p\}. \quad (6.22)$$

Then the following lemma holds:

Lemma 6.2. *For any $n \in \{1, \dots, N\}$, the set of the elementary operators dressed by the shift operator U_n :*

$$U_n \mathcal{E}_{(k,k_0)^{\mathbf{e}_N}, (a_1, k_1), \dots, (a_r, k_r)}^{(\alpha_1, \dots, \alpha_r)} U_n^{-1}, \quad (6.23)$$

is a basis in the space of the local operators at the quantum site n .

Proof. The space of the local operators in site n is generated by u_n^k and v_n^k for $k \in \{1, \dots, p-1\}$. Proceeding as done in Proposition 5.2 we have in particular the possibility to show that an alternative local basis is defined by the operators:

$$u_n^k = U_n (B^{-1}(\mu_{n,+})A(\mu_{n,+}))^k U_n^{-1}, \quad (6.24)$$

$$\tilde{\beta}_{k,n} = U_n (B^{-1}(\mu_{n,+})A(\mu_{n,+}))^k B^{-1}(\mu_{n,-})A(\mu_{n,-}) (B^{-1}(\mu_{n,+})A(\mu_{n,+}))^{p-1-k} U_n^{-1} \quad (6.25)$$

for $k \in \{1, \dots, p-1\}$. Then to prove the lemma we just have to show that the above operators are linear combinations of those defined in (6.21). Note that for $\lambda^p \neq Z_a$ with $a \in \{1, \dots, [N]\}$, the operator $B^{-1}(\lambda)$ is invertible and by the centrality of the average values we can write:

$$B^{-1}(\lambda)A(\lambda) = \frac{B(\lambda q^{p-1})B(\lambda q^{p-2}) \dots B(\lambda q)A(\lambda)}{\mathcal{B}(\Lambda)}. \quad (6.26)$$

Now the operator $B(\lambda q^{p-1})B(\lambda q^{p-2}) \cdots B(\lambda q)A(\lambda)$ is an even Laurent polynomial of degree:

$$(p-1)[N] + N - 1 + e_N = \begin{cases} pN - 1 & \text{for } N \text{ odd} \\ p(N-1) + 1 & \text{for } N \text{ even} \end{cases}$$

in λ . So for N odd to completely characterize it we have to fix its value in pN distinguished points and we are free to chose these points coinciding with the zeros of the operator $B(\lambda)$. For N even, we have to add to the B -zeros the values at the infinity, so that by using the corresponding interpolation formula for $B(\lambda q^{p-1})B(\lambda q^{p-2}) \cdots B(\lambda q)A(\lambda)$, we derive:

$$B^{-1}(\lambda)A(\lambda) = \frac{e_N}{\eta_N} \left(\frac{\lambda\Theta}{\eta_A} + \frac{\eta_A}{\lambda\Theta} \right) + \frac{1}{\eta_N^{e_N}} \sum_{a=1}^{[N]} \sum_{k=0}^{p-1} \frac{\mathcal{O}_{a,k}}{(\lambda/\eta_a^{(k)} - \eta_a^{(k)}/\lambda)}. \quad (6.27)$$

From the previous formula and the representation (6.24)-(6.25), we have that the local operators u_n^k and $\tilde{\beta}_{k,n}$ are linear combinations of the monomials $U_n \eta_N^{-e_N h} \left(\frac{\Theta}{\eta_A} \right)^{e_N h_0} \mathcal{O}_{a_1, h_1} \cdots \mathcal{O}_{a_s, h_s} U_n^{-1}$ for $s \leq p$, $a_i \in \{1, \dots, [N]\}$ and $h, h_i \in \{0, \dots, p-1\}$. For any monomial $\mathcal{O}_{a_1, h_1} \cdots \mathcal{O}_{a_s, h_s}$ we can use the commutation rules (6.18) to rewrite it in a way that operators with the same index a are adjacent, we can order them in a way that $a_i < a_j$ for $i < j \in \{1, \dots, [N]\}$ and we can apply the rule (6.15) to say if the monomial is zero or not. Finally, by using the property (6.16), we have:

$$\mathcal{O}_{a,k}^{(p+\alpha)} = \frac{\mathcal{A}(Z_a)}{\prod_{b \neq a, b=1}^N (Z_a/Z_b - Z_b/Z_a)} \mathcal{O}_{a,k}^{(\alpha)}, \quad (6.28)$$

and so it is clear that all the non-zero monomials $\mathcal{O}_{a_1, h_1} \cdots \mathcal{O}_{a_s, h_s}$ can be written in the form (6.21). \square

6.2.2 Form factors of elementary operators

As anticipated the interest in the above definition of elementary operators is the simplicity of their form factors:

Lemma 6.3. *Let $\langle t_k |$ and $|t'_{k'}\rangle$ be two eigenstates of the transfer matrix $T(\lambda)$, then it holds:*

$$\langle t_k | \mathcal{E}_{(h, h_0)^{e_N}, (a_1, h_1), \dots, (a_r, h_r)}^{(\alpha_1, \dots, \alpha_r)} | t'_{k'} \rangle = \frac{\delta_{k, k' + h}^{e_N} q^{e_N h_0 k'}}{\eta_N^{(0)h e_N} \prod_{j=1}^N \xi_j^{e_N h_0}} f_{(e_N h_0, \{\alpha\}, \{a\})} \det_{[N] + rp - g} (||O_{a,b}^{(e_N h_0, \{\alpha\}, \{a\})}||), \quad (6.29)$$

where $||O_{a,b}^{(e_N h_0, \{\alpha\}, \{a\})}||$ is the $([N] + rp - g) \times ([N] + rp - g)$ matrix of elements:

$$O_{a, \sum_{h=1}^{m-1} (p - \alpha_h + 1) + j_m}^{(e_N h_0, \{\alpha\}, \{a\})} \equiv \left(\eta_{a_m}^{(h_m + j_m)} \right)^{4(a-1)} \quad \text{for } j_m \in \{0, \dots, p - \alpha_m\}, \quad m \in \{1, \dots, r\}, \quad (6.30)$$

$$O_{a, \sum_{h=1}^r (p - \alpha_h + 1) + i}^{(e_N h_0, \{\alpha\}, \{a\})} \equiv \Phi_{b_i, a + (e_N h_0 + g)/2}^{(t, t')} \quad \text{for } i \in \{1, \dots, [N] - r\}, \quad g \equiv \sum_{h=1}^r \alpha_h, \quad (6.31)$$

for any $a \in \{1, \dots, [N] + rp - g\}$. Here, we have defined $\{b_1, \dots, b_{[N]-r}\} \equiv \{1, \dots, [N]\} \setminus \{a_1, \dots, a_r\}$ with elements ordered by $b_i < b_j$ for $i < j$ and

$$\begin{aligned} f_{(\mathbf{e}_N h_0, \{\alpha\}, \{a\})} &\equiv \frac{\prod_{i=1}^r Q_{t'}(\eta_{a_i}^{(h_i-\alpha_i)}) \bar{Q}_t(\eta_{a_i}^{(h_i)}) \frac{(\eta_{a_i}^{(h_i)})^{\mathbf{e}_N h_0 + \alpha_i ([N]-r)}}{\omega_{a_i}(\eta_{a_i}^{(h_i)})} \prod_{h=0}^{\alpha_i-1} a(\eta_{a_i}^{(h_i-h)})}{\prod_{i=1}^r \prod_{h=0}^{\alpha_i-1} \prod_{j=1}^{i-1} \left(\frac{\eta_{a_i}^{(h_i+\alpha_i-h)}}{\eta_{a_j}^{(h_j)}} - \frac{\eta_{a_j}^{(h_j)}}{\eta_{a_i}^{(h_i+\alpha_i-h)}} \right) \prod_{j=i+1}^r \left(\frac{\eta_{a_i}^{(h_i)}}{\eta_{a_j}^{(h_j+h)}} - \frac{\eta_{a_j}^{(h_j+h)}}{\eta_{a_i}^{(h_i)}} \right)} \\ &\times \frac{(-1)^{\sum_{i=1}^r (a_i-i)} \prod_{i=1}^r q^{-([N]-r)\alpha_i(\alpha_i-1)/2} V((\eta_{a_1}^{(h_1)})^2, \dots, (\eta_{a_r}^{(h_r)})^2)}{\prod_{i=1}^r \prod_{j=1}^{[N]-r} (Z_{a_i}^2 - Z_{b_j}^2) V((\eta_{a_1}^{(h_1)})^2, \dots, (\eta_{a_1}^{(h_1+p-\alpha_1)})^2, \dots, (\eta_{a_r}^{(h_r)})^2, \dots, (\eta_{a_r}^{(h_r+p-\alpha_r)})^2)}), \end{aligned} \quad (6.32)$$

where $V(x_1, \dots, x_N) \equiv \prod_{1 \leq b < a \leq N} (x_a - x_b)$ is the Vandermonde determinant.

Proof. The operator $\eta_N^{-\mathbf{e}_N h} \left(\frac{\Theta}{\eta_A} \right)^{\mathbf{e}_N h_0}$ act in the following way on the state $\langle t_k |$:

$$\begin{aligned} \langle t_k | \eta_N^{-\mathbf{e}_N h} \left(\frac{\Theta}{\eta_A} \right)^{\mathbf{e}_N h_0} &= \\ &= \left(\frac{q^{h_0(k-h)}}{(\eta_N^{(0)})^2 \prod_{j=1}^N \xi_j^{h_0}} \right)^{\mathbf{e}_N} \sum_{k_1, \dots, k_N=1}^p \left(\frac{q^{(k-h)h_N}}{p^{1/2}} \right)^{\mathbf{e}_N} \prod_{a=1}^{[N]} (\eta_a^{(k_a)})^{h_0 \mathbf{e}_N} \bar{Q}_t(\eta_a^{(k_a)}) \\ &\times \prod_{1 \leq b < a \leq [N]} ((\eta_a^{(k_a)})^2 - (\eta_b^{(k_b)})^2) \frac{\langle \eta_1^{(k_1)}, \dots, \eta_N^{(k_N)} |}{\prod_{b=1}^{[N]} \omega_b(\eta_b^{(k_b)})}. \end{aligned} \quad (6.33)$$

From the formula (6.19), it follows:

$$\langle \eta_1^{(k_1)}, \dots, \eta_{a_i}^{(f)}, \dots, \eta_N^{(k_N)} | \mathcal{O}_{a_i, h_i}^{(\alpha_i)} = \frac{\prod_{h=0}^{\alpha_i-1} a(\eta_{a_i}^{(h_i-h)}) \delta_{f, h_i} \langle \eta_1^{(k_1)}, \dots, \eta_{a_i}^{(h_i-\alpha_i)}, \dots, \eta_N^{(k_N)} |}{\prod_{b \neq a_i, b=1}^{[N]} \prod_{h=0}^{\alpha_i-1} (\eta_{a_i}^{(h_i-h)} / \eta_b^{(k_b)} - \eta_b^{(k_b)} / \eta_{a_i}^{(h_i-h)})}. \quad (6.34)$$

So we can compute also the action of $\mathcal{O}_{a_1, h_1}^{(\alpha_1)} \dots \mathcal{O}_{a_r, h_r}^{(\alpha_r)}$ just taking into account the order of the operators in the monomial which leads by the scalar product formula to:

$$\begin{aligned} \langle t_k | \mathcal{E}_{(h, h_0)^{\mathbf{e}_N}, (a_1, h_1), \dots, (a_r, h_r)}^{(\alpha_1, \dots, \alpha_r)} | t'_{k'} \rangle &= \left(\frac{q^{h_0(k-h)}}{(\eta_N^{(0)})^h \prod_{j=1}^N \xi_j^{h_0}} \right)^{\mathbf{e}_N} \sum_{k_1, \dots, k_N=1}^p \left(\frac{q^{[(k-h)-k']k_N}}{p} \right)^{\mathbf{e}_N} \prod_{a=1}^{[N]} (\eta_a^{(k_a)})^{h_0 \mathbf{e}_N} \\ &\times \prod_{i=1}^r \frac{\prod_{h=0}^{\alpha_i-1} a(\eta_{a_i}^{(h_i-h)}) \delta_{k_{a_i}, h_i}}{\prod_{j=1}^{[N]-r} \prod_{h=0}^{\alpha_i-1} (\eta_{a_i}^{(h_i-h)} / \eta_{b_j}^{(k_{b_j})} - \eta_{b_j}^{(k_{b_j})} / \eta_{a_i}^{(h_i-h)})} \\ &\times \prod_{i=1}^r \prod_{h=0}^{\alpha_i-1} \frac{\prod_{j=i+1}^r (\eta_{a_i}^{(h_i-h)} / \eta_{a_j}^{(h_j)} - \eta_{a_j}^{(h_j)} / \eta_{a_i}^{(h_i-h)})^{-1}}{\prod_{j=1}^{i-1} (\eta_{a_i}^{(h_i-h)} / \eta_{a_j}^{(h_j-\alpha_j)} - \eta_{a_j}^{(h_j-\alpha_j)} / \eta_{a_i}^{(h_i-h)})} \\ &\times \prod_{j=1}^{[N]-r} \frac{Q_{t'}(\eta_{b_j}^{(k_{b_j})}) \bar{Q}_t(\eta_{b_j}^{(k_{b_j})})}{\omega_{b_j}(\eta_{b_j}^{(k_{b_j})})} \prod_{i=1}^r \frac{Q_{t'}(\eta_{a_i}^{(h_i-\alpha_i)}) \bar{Q}_t(\eta_{a_i}^{(h_i)})}{\omega_{a_i}(\eta_{a_i}^{(h_i)})} \\ &\times V((\eta_1^{(h_1)})^2, \dots, (\eta_{[N]}^{(h_{[N]})})^2). \end{aligned} \quad (6.35)$$

Let us remark that the sum $\sum_{k_1, \dots, k_N=1}^p$ reduces to $\delta_{k, k'+h}^{\mathbf{e}_N}$ times the sum $\sum_{k_{b_1}, \dots, k_{b_{[N]-r}}=1}^p$ for the presence of the $\prod_{i=1}^r \delta_{k_{a_i}, h_i}$. Now we multiply each term of the sum by:

$$1 = \prod_{\epsilon=\pm 1} \prod_{i=1}^r \prod_{j=1}^{[N]-r} \prod_{h=-p+\alpha_i}^{-1} ((\eta_{a_i}^{(h_i-h)})^2 - (\eta_{b_j}^{(k_{b_j})})^2)^\epsilon \times \left(\frac{V((\eta_{a_1}^{(h_1)})^2, \dots, (\eta_{a_1}^{(h_1+p-\alpha_1)})^2, \dots, (\eta_{a_r}^{(h_r)})^2, \dots, (\eta_{a_r}^{(h_r+p-\alpha_r)})^2)}{V((\eta_{a_1}^{(h_1)})^2, \dots, (\eta_{a_r}^{(h_r)})^2)} \right)^\epsilon \quad (6.36)$$

here the power $+1$ leads to the construction of the Vandermonde determinant:

$$V(\underbrace{(\eta_{a_1}^{(h_1)})^2, \dots, (\eta_{a_1}^{(h_1+p-\alpha_1)})^2}_{p-\alpha_1+1 \text{ columns}}, \dots, \underbrace{(\eta_{a_r}^{(h_r)})^2, \dots, (\eta_{a_r}^{(h_r+p-\alpha_r)})^2}_{p-\alpha_r+1 \text{ columns}}, \underbrace{(\eta_{b_1}^{(k_{b_1})})^2, \dots, (\eta_{b_{[N]-r}}^{(k_{b_{[N]-r}})})^2}_{[N]-r \text{ columns}}), \quad (6.37)$$

and the sum $\sum_{k_{b_1}, \dots, k_{b_{[N]-r}}=1}^p$ becomes sum over columns and after some algebra we get our formula. \square

It is interesting to point out that the last $[N] - r$ columns of the matrix $||O_{a,b}^{\mathbf{e}_N h_0, \{\alpha\}, \{a\}}||$ are just those of the scalar product.

Remark 7. For the similarity of the model and representations considered, it is natural to cite the series of works [143–146]. There, in the framework of cyclic SOV-representations, first results on the matrix elements of local operators appear. However, it is worth saying that these quantities are there computed only for the restriction¹⁷ of the τ_2 -model to the generalized Ising model. In particular, the matrix elements of u_1 are computed and the results are not presented in a determinant form.

7 Conclusion and outlook

7.1 Results

In this article we have considered the lattice sine-Gordon model in cyclic representations and we have solved in this case two fundamental problems for the computation of matrix elements of local operators:

- Scalar products: determinant of $N \times N$ matrices whose matrix elements are sums over the spectrum of each quantum separate variable of the product of the coefficients of states, this being for all the left/right separate states in the SOV-basis.
- Inverse problem solution: reconstruction of all local operators in terms of standard Sklyanin's quantum separate variables.

Further, we have shown how these results lead to the computation of matrix elements of all local operators. At first, standard¹⁸ Sklyanin's quantum separate variables are suitable for solving the transfer matrix spectral

¹⁷It can be compared to the restriction to the case $q^2 = 1$ for the even sine-Gordon chain.

¹⁸Coinciding with the operator-zeros of one of the Yang-Baxter algebra generators, like $B(\lambda)$ or $C(\lambda)$.

problem. Indeed, the transfer matrix spectrum (eigenvalues & eigenstates) admits a simple and complete characterization in terms of Baxter-equation solutions in this SOV-basis. Then the inverse problem solution allows to write the action of any local operator on transfer matrix eigenstates as finite sums of separate states in the SOV-basis. Hence, the matrix elements of any local operator are written as finite sums of determinants of the resulting scalar product formulae.

We have explicitly developed this program characterizing the matrix elements of the local operators u_n and α_n by one determinant formulae in terms of matrices obtained by modifying a single row in the scalar product matrices. Moreover, we have constructed an operator basis whose matrix elements are in turn written by one determinant formulae. The matrices involved have rows which coincide with those of the scalar product matrix or with those of the Vandermonde matrix computed in the spectrum of the separate variables.

7.2 Comparison with previous SOV-results

In the literature of quantum integrable models there exist several results on matrix elements of local operators which can be traced back to applications of separation of variable methods. In this section, we try to recall the most relevant ones as they allow for an explicit comparison with our results. It leads to a universal picture emerging in the characterization of matrix elements by SOV-methods.

7.2.1 On the reconstruction of local operators

One important motivation for our work was to introduce a well defined setup which allows to solve the longstanding problem of the identification of local operators in the continuum sine-Gordon model thanks to the reconstructions achieved on the lattice. Then, it should also allow for the identification of form factor solutions of the continuous theory by implementing well defined limits from our lattice formulae.

Even if methodologically different, it is worth recalling the semi-classical reconstruction presented by Babelon, Bernard and Smirnov in [93] for chiral local operators of the restricted sine-Gordon model (in the infinite volume) at the reflectionless points, $\beta^2 = 1/(1+\nu)$ with $\nu \in \mathbb{Z}^{\geq 0}$. The classical sine-Gordon model admits a SOV description: each n -soliton solution $\varphi(x, t)$ of the equation of motion can be represented in terms of n -separate variables A_j , which in the BBS choice [93] lead to the representations:

$$e^{i\varphi} = \prod_{j=1}^n \frac{A_j}{B_j}, \quad (7.1)$$

where the B_j are integrals of motion. The formula (7.1) represents classically a SOV reconstruction of the local fields when restricted to the n -soliton sector. In [93], this reconstruction has been extended to the quantum model in each n -solitons sector by quantizing the separate variables¹⁹ A_j and the conjugate momenta as operators which generate n independent Weyl algebras with parameter $\tilde{q} = e^{i\pi \frac{\beta^2}{1-\beta^2}} = e^{i\frac{\pi}{\nu}}$. This extension and the consequent identifications of primary fields and their chiral descendants in the perturbed

¹⁹A fundamental point is the introduction of appropriate Hermitian conjugation properties and the characterization of the spectrum of the Weyl algebra generators.

minimal models $M_{1,1+\nu}$ are justified by the following indirect but strong arguments: a) The n -multiple integrals of the form (36) in [93] which represent the n -solitons to n -solitons form factors²⁰ of chiral left operators at the reflectionless points are reproduced from the semi-classical limit. b) The counting of these form factor solutions allows the reconstruction of the chiral characters of $M_{1,1+\nu}$ [94]. Further support to (7.1) was given by Smirnov's work on semi-classical form factors²¹ of the continuous KdV model in finite volume [97]; there the form factors of [93] were reproduced by taking the infinite volume limit of the KdV semi-classical ones.

Let us remark that in our lattice regularization of the sine-Gordon model, choosing as quantum separate variables standard²² Sklyanin's ones, the reconstruction of the exponential fields is not of the simple form given in (7.1). Then, the following question is relevant: is it possible to find a SOV representation of the quantum lattice sine-Gordon model where the exponential fields are simply written as a product of generators of the SOV representations?

A natural idea can be to implement a change of basis in the quantum separate variables from Sklyanin's ones to a new set; an interesting example of this approach was used by Babelon in [99] which has provided a simple reconstruction of the lattice quantum Toda local operators in terms of a set of quantum separate variables defined by a change of variables from the Sklyanin's ones. However, it is worth pointing out that, for a new SOV representation to be really useful for the computation of matrix elements, it should not only give a simple reconstruction of the local operators but also keep the solution of the transfer matrix spectral problem and the scalar product formulae as simple as for original Sklyanin's variables. Let us comment that a reconstruction like (7.1) can be formally derived at the quantum level implementing the special limit

$$\kappa_n/i \rightarrow +\infty \quad (7.2)$$

on the following reconstruction formula of the lattice sine-Gordon model:

$$\frac{(q^{-1}v_n^2 + \kappa_n^2)}{(q^{-1}v_n^2 \kappa_n^2 + 1)} = U_n A^{-1}(\mu_{n,-}) B(\mu_{n,-}) B^{-1}(\mu_{n,+}) A(\mu_{n,+}) U_n^{-1}. \quad (7.3)$$

The result for an even chain reads

$$v_n^{-2h} = \frac{\Theta^{2h}}{\prod_{a \neq n, a=1}^N \xi_a^{2h}} U_n \prod_{a=1}^{N-1} \eta_a^{2h} U_n^{-1}, \quad h \in \{1, \dots, p-1\} \quad (7.4)$$

in terms of Sklyanin's separate variables. It is possible to argue that the previous limit can be consistently interpreted as a chiral deformation of the lattice sine-Gordon model to chiral KdV models [134, 153]. In a future publication, we will analyze the cyclic representations²³ of these chiral KdV models showing our statement on the reconstruction formulae and computing the matrix elements. It is worth pointing out that the form factors of the r.h.s. of (7.4) are trivial to compute in our SOV framework and are written by one determinant formulae which differ w.r.t. the scalar product only for $h \in \{1, \dots, p-1\}$ rows.

²⁰Note that these are solutions of the form factor equations and so they surely represent local fields in the S-matrix formulation of the restricted sine-Gordon model.

²¹On the basis of this last Smirnov's work, Lukyanov has introduced his conjecture for the finite temperature expectations values of exponential fields in finite volume for the shG-model [98].

²²The same is true if we take the products of the operator zeros of $C(\lambda)$ but also of $A(\lambda)$ and $D(\lambda)$, i.e. for all possibilities to construct the SOV representations by the simplest Sklyanin's method.

²³Here, we are referring to compact representations of chiral KdV models where the generators of the local Weyl algebras are unitary operators. The spectrum of the non-compact versions was instead analyzed by SOV and Q-operator method in [102].

7.2.2 On the matrix elements of local operators

In the case of the quantum integrable Toda chain [88], Smirnov [97] has derived in the framework of Sklyanin's SOV determinant formulae for the matrix elements of a conjectured basis of local operators which look very similar to our formulae. The main difference is due to the different nature of the spectrum of the quantum separate variables in the two models. In fact, in the case of the lattice Toda model, Sklyanin's measure is continuous (continuous SOV-spectrum) while it is discrete in the case of the cyclic lattice sine-Gordon model. The elements of the matrices whose determinants give the form factor formulae are then expressed as "convolutions", over the spectrum of the separate variables, of Baxter equations solutions plus contributions coming from the local operators. In the case of Smirnov's formulae they are true integrals, the SOV-spectrum being continuous, while in our formulae they are "discrete convolutions", the SOV-spectrum being discrete. Let us comment that the need to conjecture²⁴ the form of a basis of local operators in [97] is due to the lack of a direct reconstruction of local operators in terms of Sklyanin's separate variables.

In the case of the infinite volume quantum sine-Gordon field theory, the form factors of local operators [67] have also a form similar to the one predicted by SOV. This similarity can be made explicit considering the n -soliton form factors for the restricted sine-Gordon theory at the reflectionless points in formula (31) of [93]. Then, for the local fields interpreted as primary operators in [93], the corresponding form factors can be easily rewritten as determinants of $n \times n$ matrices whose elements are integral convolutions of n -soliton wave functions (the ψ -functions (32)) plus contributions coming from the local operators.

The rough picture that seems to emerge is that by performing the IR limit on our lattice form factors the lattice wave functions factorized in terms of Q-operator eigenvalues have to converge to the infinite volume n -soliton wave functions. To which extent this picture can be confirmed and clarified by a detailed analysis of the thermodynamic limit starting from our lattice sine-Gordon model results is of course an interesting question to which we would like to answer in the future.

7.3 Outlook

It is worth mentioning that we didn't succeed yet to express the matrix elements of discretized exponential of the sine-Gordon field in terms of one simple determinant formula. Hence our next natural project is the simplification of the present representation; this is also important in view of the attempt to extend our results from the lattice to the continuous finite and infinite volume limits. The main goal here is to derive the known form factors of the IR limit, starting from our lattice form factors, in this way solving the longstanding problem of the identifications of local fields in the S-matrix characterization of the infinite volume sine-Gordon model.

Beyond the sine-Gordon model we want to point out the potential generality of the method we have introduced here to compute matrix elements of local operators for quantum integrable models. The main ingredients used to develop it are the reconstruction of local operators in Sklyanin's SOV representations and the scalar product formulae for the transfer matrix eigenstates (and general separate states). The emerging picture is the possibility to apply this method to a whole class of integrable quantum model which were

²⁴The consistency of this conjecture is there verified by a counting argument based on the existence of an appropriate set of null conditions for the "integral convolutions".

not tractable with other methods. This is in particular the case for lattice integrable quantum models to which the algebraic Bethe ansatz does not apply. The first remarkable case is given by the τ_2 -model in general representations which are of special interest for their connection to the chiral Potts model. This will be the next model that we will analyze by our technique due to the similarity of its cyclic representations with those of the sine-Gordon model.

There are also many other examples which are interesting and for which, on the one hand, the reconstruction of the local operators can be deduced from [29] and, on the other hand, the description of the spectrum can be given by Sklyanin's quantum separation of variables. For all these models the possibility to apply our method for the computation of matrix elements is very concrete and moreover the results are expected to have a completely similar form to the ones shown in the present article.

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